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GENERAL EDITORS

R. H. FOWLER AND P. KAPITZA

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THE QUANTUM THEORY OF RADIATION

BY

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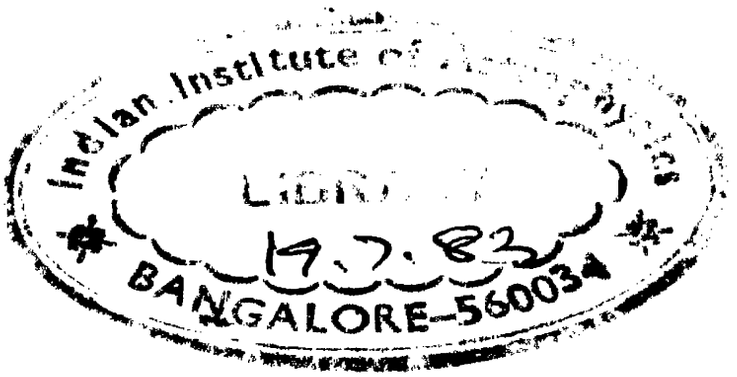
SECOND EDITION



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PREFACE TO THE SECOND EDITION

ALTHOUGH it is with a good conscience that I allow the bulk of the book to be reprinted in practically the old form I would have wished to amplify it considerably and to rewrite a few sections. War conditions have made this impossible. New matter could only be added at the end of the book and to a limited extent. The reader will find new sections on the cascade theory of showers and on the general theory of damping. By enlarging the appendix I have tried to include a few of the more important topics which otherwise would have been treated in the text.

DUBLIN, W. H.
March, 1944

FROM THE PREFACE TO THE FIRST EDITION

THE idea of writing this book occurred to me while giving a course of lectures on radiation theory in Göttingen in 1932–3. In the absence of any comprehensive treatise on the subject, it seemed to me desirable to present the theory as far as possible from a uniform point of view. This is here attempted.

It has not, of course, been possible in a volume of this size to deal exhaustively with all applications, but I have endeavoured to treat all those of major importance.

The references make no claim to completeness. This applies particularly to the experimental papers, which were selected more or less arbitrarily with the view to checking at each stage the results of the theory.

I am greatly indebted to Professor N. F. Mott who has read the whole work in manuscript and to Mr. K. Fuchs who carried out many of the numerical calculations and helped me in reading the proofs. For this I wish to express to them my deepest thanks. My thanks are also due to Dr. H. R. Hulme for useful criticism.

BRISTOL, W. H.
November, 1935

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INTRODUCTION

In the historical development of theoretical physics the theory of radiation plays a particular role. It was actually in the theory of radiation that the break-down of classical concepts first became evident, and it was the problem of the thermal radiation of a black body that led Planck in 1900 to introduce the quantum of action h. This concept gave the impetus to a rather rapid development, in which the landmarks are Einstein's hypothesis of light quanta (1905), Bohr's first application of the quantum theory to atomic physics (1913), and finally the foundation of quantum mechanics (1925–6).

Through the development of quantum mechanics, the whole field of atomic and molecular physics has become susceptible to quantitative theoretical discussion. The progress, however, has been more striking in atomic mechanics than in the theory of radiation itself; for when the principles which had been applied successfully in quantizing the motion of an electron were transferred to the problem of the electromagnetic field, difficulties soon became apparent which could not be overcome within the framework of these principles themselves owing to divergences of the integrals expressing the interaction of light with an electron in the second approximation, radiative transitions to negative energy states, etc.

At first it seemed that, in view of these difficulties, a consistent quantum theory of the electromagnetic field could not be found. The later developments of the theory, however, have shown that, with certain limitations, this is not the case. Firstly, one of these difficulties, that of the negative energy levels, has to some extent been removed by the discovery of positive electrons. Although in the theory of positive electrons new difficulties occur, and although the other difficulties are not yet fully understood (we shall discuss them in § 25), it seems now that there is a certain limited field within which the present quantum electrodynamics is *correct*.

A great advance in the critical understanding of the situation is due to an investigation by Bohr and Rosenfeld, which has made it clear that the present quantum theory of the electromagnetic field—at least *in vacuo*—forms, together with particle mechanics, a consistent scheme from which neither of the two parts can be omitted. Thus, in a sense, we may say that the quantum theory of the

field has equal rights with quantum mechanics to rank as a valid theory.

The present theory can be correctly applied to the interaction of light with elementary particles to the first approximation. The difficulties which occur, especially in the higher approximations to this interaction, seem to be rather like the 'ultra-violet catastrophe' (divergence of the radiation formula for high frequencies) in the classical theory: they show the limits within which the theory is valid. It may be hoped that the examination of these difficulties will lead in the future to a further development of the theory to a stage when it will probably comprise nuclear physics and give some information about the nature of the elementary particles.

If the application of the theory is confined within these limits, it will be seen in this book that the theory gives—qualitatively and quantitatively—a full account of the experimental facts within a large field, including even phenomena connected with the creation and annihilation of positive electrons. In applying the theory to such processes, however, it is even more necessary to consider carefully the limits of the theory.

Thus it seems that the theory is well enough developed, and its limits of application well enough marked, for a summary to be given.

In this book, after an introductory chapter on the classical theory, we shall develop the formalism of the quantum theory of radiation in the simplest form of general validity, which is certainly that given by Dirac and Fermi (expansion of the total field in plane waves). It is equivalent to the general theory of Heisenberg and Pauli.

The theory is based entirely on the correspondence principle. This has been shown, for instance, in a paper by Heisenberg in which he puts the theory in such a form that the development is exactly analogous to the classical method of treatment (Hertzian vector, etc.). This correspondence may be shown equally by writing the classical theory in such a form that it corresponds to the quantum-mechanical way of treatment (§ 6). We prefer this representation, because, for applications to atomic physics, Dirac's is certainly more convenient than the classical formalism. It will also be seen that, even for classical problems, this form of the theory is not more complicated than the usual one.

The applications to atomic physics discussed here are limited to those of fundamental interest (emission, absorption, dispersion, line

breadth, etc.). We shall not consider in detail questions relevant to atomic physics rather than to the theory of radiation, such as selection rules and the atomic scattering factor for X-rays, etc. Nor shall we treat in great detail the older parts of the subject (Planck's law) which are adequately discussed in many text-books.

We shall, however, pay special attention to processes which occur at high energies (Compton scattering, positive electron formation, etc.) as it seems that this field is the most important for development in the near future. Also the problem of the reaction of the field on the electron will be considered in detail in both the classical and the quantum theory.

The experiments quoted are those which are capable of testing definite points of the theory, even if their accuracy is not very high. Naturally only some of the important experiments dealing with our subject can be considered.

The results of the theory are evaluated numerically, as far as is possible in a volume of limited size. In this respect, also, the region of high energies has been considered in greater detail than the optical region. We hope that the tables and graphs in which these results are presented may also be useful to experimental physicists working in this field.

In the years since this book was written the general view taken, namely that quantum electrodynamics is, within certain limits, a correct theory, has been enhanced in several ways. The discovery of cascade showers has shown that the theory remains valid up to the highest energies known, contrary to what was formerly believed. Furthermore, it has been possible to comprise the useful parts of the theory into a general and simple mathematical scheme which is free of divergencies and which actually goes somewhat beyond the use of the first approximation in the interaction between the field and the particles. It also includes a general account of the reaction of the field on the elementary particles to the extent in which the latter has classically an unambiguous meaning for point charges. Thus the limits of the theory are clearly drawn, but the difficulties have only been removed from its surface, not from its foundations. New attempts have been made on several lines (Born-Infeld, Dirac, Born-Peng) but none of them is as yet developed far enough for a final judgement to be passed. They are not described in detail in this book.

I

CLASSICAL THEORY OF RADIATION

1. The general Maxwell-Lorentz theory

1. *Field equations.* THE classical theory of radiation is based on Maxwell's theory of the electromagnetic field. The two fundamental quantities describing the electromagnetic field are the electrical and magnetic field strengths **E** and **H**, which are both functions of space and time. To describe the electric state of matter one needs, besides the field, the charge density ρ and current density **i** which are also functions of space and time. If the velocity of the charge at a given point and a given time is **v**, the current density is

$$\mathbf{i} = \rho\mathbf{v}. \tag{1}$$

For a given distribution of charge and current the field is determined by the *Maxwell-Lorentz equations*

$$\operatorname{curl}\mathbf{E} + \frac{1}{c}\dot{\mathbf{H}} = 0 \tag{2a}$$

$$\operatorname{div}\mathbf{H} = 0 \tag{2b}$$

$$\operatorname{curl}\mathbf{H} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}\rho\mathbf{v} \tag{2c}$$

$$\operatorname{div}\mathbf{E} = 4\pi\rho. \tag{2d}$$

(The point denotes differentiation with respect to the time t.)

From these equations it can easily be deduced that charge and current satisfy the equation of continuity (conservation of charge)

$$\operatorname{div}(\rho\mathbf{v}) + \dot{\rho} = 0. \tag{3}$$

On the other hand, the motion of the charges in a given field is determined by the *Lorentz equation*

$$\mathbf{k} = \rho\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{H}]\right), \tag{4}$$

where **k** represents the *density of force* acting on the charge density ρ. This electromagnetic force is in equilibrium with the force of inertia, which is given by the mass distribution of the charges.

For a point charge e (elementary particle) we have to carry out in the equations (2) and (4) the transition to the case where ρ is concentrated in an infinitely small volume. The Lorentz equation (4)

can then be integrated over this volume, giving the whole force acting on the particle

$$\mathbf{K} = e\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right). \tag{5}$$

$\mathbf{K}$ has to be equated to the force of inertia

$$\mathbf{K} = \frac{d}{dt}(m\mathbf{v}), \tag{6}$$

m being the inert mass of the particle.

The field which has to be inserted in the Lorentz equation (4) or (5) is the external field produced by other charges (condensers, magnets, etc.) as well as the field produced by the point-charge itself. This self-produced field will also react on the motion of the particle. But fortunately, this reaction of the field is in general very small, so that we may insert in the equation of motion the external field only. The theory of the reaction of the field on the point-charge, which is connected with the problem of the inertial mass of the particle, is faced with great difficulties and at present has been developed only in an approximate way. We shall discuss it later in § 4.

2. *Potentials.* The field equations (2) can be reduced to simpler equations between a vector and a scalar function only instead of two vectors. From equation (2 b) it follows that $\mathbf{H}$ can always be represented as a rotation of another vector $\mathbf{A}$, which we call the *vector potential*:

$$\mathbf{H} = \operatorname{curl}\mathbf{A}. \tag{7a}$$

Then (2 a) becomes

$$\operatorname{curl}\left(\mathbf{E} + \frac{1}{c}\dot{\mathbf{A}}\right) = 0$$

or

$$\mathbf{E} + \frac{1}{c}\dot{\mathbf{A}} = -\operatorname{grad}\phi, \tag{7b}$$

where ϕ represents a scalar function which we call the *scalar potential.* The other two equations (2 c, d) result in two differential equations for these potentials. Making use of the general vector relation curlcurl = graddiv$-\nabla^2$ they can be written in the form:

$$\frac{1}{c^2}\ddot{\mathbf{A}} - \nabla^2\mathbf{A} + \operatorname{grad}\left(\operatorname{div}\mathbf{A} + \frac{1}{c}\dot{\phi}\right) = \frac{4\pi}{c}\rho\mathbf{v} \tag{8a}$$

$$-\nabla^2\phi - \frac{1}{c}\operatorname{div}\dot{\mathbf{A}} = 4\pi\rho. \tag{8b}$$

The vector **A** is not determined completely by the magnetic field **H**. Since, for any scalar function χ, curlgrad $\chi = 0$, we can add to **A** the gradient of an arbitrary function χ. According to (7 b), however, we have to replace ϕ by $\phi + \frac{1}{c}\dot{\chi}$ if we replace **A** by **A**—grad χ, in order that **E** should not be changed. This freedom in the choice of the potentials can be used to simplify the field equations (8). If $\mathbf{A}_0$ and ϕ_0 represent certain possible values of **A** and ϕ, we determine χ from the equation

$$\nabla^2\chi - \frac{1}{c^2}\ddot{\chi} = \operatorname{div}\mathbf{A}_0 + \frac{1}{c}\dot{\phi}_0. \tag{9}$$

If we now put

$$\mathbf{A} = \mathbf{A}_0 - \operatorname{grad}\chi,$$

$$\phi = \phi_0 + \frac{1}{c}\dot{\chi},$$

we obtain

$$\operatorname{div}\mathbf{A} + \frac{1}{c}\dot{\phi} = 0. \tag{10}$$

(10) represents a relation between the potentials and is called the *Lorentz relation*. The field equations (8) then become simply

$$\frac{1}{c^2}\ddot{\mathbf{A}} - \nabla^2\mathbf{A} = \frac{4\pi}{c}\rho\mathbf{v}, \tag{11a}$$

$$\frac{1}{c^2}\ddot{\phi} - \nabla^2\phi = 4\pi\rho. \tag{11b}$$

A and ϕ satisfy therefore the *inhomogeneous wave equation*. They are coupled by the Lorentz condition (10) only.

Still **A** and ϕ are not yet determined completely by the field strengths **E** and **H**. χ is limited in so far as equation (9) has to be satisfied. We are still free to choose an arbitrary χ which satisfies the homogeneous wave equation

$$\nabla^2\chi - \frac{1}{c^2}\ddot{\chi} = 0. \tag{12}$$

Replacing **A** by **A**—grad χ and ϕ by $\phi + \dot{\chi}/c$ the field strengths and the Lorentz condition (10) remain unchanged. The invariance under this transformation is called 'gauge invariance'.

3. *Retarded Potentials.* The general solutions of the wave equations (11) can easily be written down. As is well known, a special solution

of the Poisson equation $\nabla^2\phi = 4\pi\rho$ is represented by the Newtonian potential

$$\phi(P) = \int \frac{\rho(P')\, d\tau'}{r_{PP'}},$$

where the integration has to be carried out all over the space. $r_{PP'}$ is the distance between a point P' of the charge distribution and the point P at which we are evaluating the potential ϕ. From this solution one can easily find also a special solution of the time dependent Poisson equation (11b):

$$\phi(P,t) = \int \frac{\rho(P', t - r_{PP'}/c)}{r_{PP'}}\, d\tau'. \tag{13a}$$

This expression has the following meaning: If we wish to know the potential at a point P at a time t, we have to take for each point P' of the space, the charge density which was present there at the previous time $t - r_{PP'}/c$. Therefore for each point of the integration space we have to take the density at another time. $t - r_{PP'}/c$ represents the time which the light needs to come from P' to the point P at which the potential is considered. Therefore (13a) takes account of the finite velocity c of propagation of an electromagnetic field. (13a) can also be interpreted in the following way: We consider a spherical light wave which is contracting with the velocity c, so that it arrives at the origin P at the time t. During its path this light wave meets at each point P' of space a certain charge density $\rho(P', t - r_{PP'}/c)$. It then collects all the contributions from all points of space and delivers them at P (after division by $r_{PP'}$) giving the potential $\phi(P,t)$.

In the same way we obtain a special solution of (11a)

$$\mathbf{A}(P,t) = \frac{1}{c} \int \frac{(\rho\mathbf{v})(P', t - r_{PP'}/c)}{r_{PP'}}\, d\tau'. \tag{13b}$$

The potentials (13) are called the *retarded potentials*. The Lorentz condition (10) is obviously satisfied by the solution (13) because of the conservation of charge (3).

(13) represents a special solution of the field equations, namely the field which arises from the considered charges only. To obtain the general solution we have to add the general solution of the homogeneous wave equations

$$\left.\begin{aligned} \nabla^2\mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} &= 0 \\ \nabla^2\phi - \frac{1}{c^2}\ddot{\phi} &= 0 \\ \operatorname{div}\mathbf{A} + \frac{1}{c}\dot{\phi} &= 0, \end{aligned}\right\} \tag{14}$$

representing the field in the free space.

For this part of the field, which satisfies the homogeneous wave equations (14) one can, according to (12), chose χ so that the scalar potential ϕ vanishes. The field which is independent of the charges is therefore simply given by

$$\left.\begin{aligned} \nabla^2\mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} &= 0, \qquad \operatorname{div}\mathbf{A} = 0 \\ \mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}}, &\qquad \mathbf{H} = \operatorname{curl}\mathbf{A}. \end{aligned}\right\} \tag{14'}$$

The general solution of (14′) is formed by superposing *transverse waves* (see § **6**).

The retarded potentials (13) will be evaluated in § **3** for the case of an arbitrary motion of a point charge and then be applied to problems of the emission of light, etc.

4. *Energy and momentum balance.* It is assumed in Maxwell's theory that a volume in which the field is different from zero contains a certain amount of energy and momentum. The energy in a given volume is given by:

$$U = \frac{1}{8\pi}\int (E^2 + H^2)\, d\tau = \int u\, d\tau. \tag{15}$$

The density of momentum is assumed to be $1/c^2$ times the energy passing through unit area per unit time. The latter is given by the Poynting vector $\mathbf{S}$.

It is convenient, in all discussions of high energy radiation, to introduce the *quantity* $c \times$ *momentum*, having the dimensions of energy. Therefore, throughout this book, we shall *refer to the latter quantity simply as 'momentum'*, for particles and for radiation.

According to this definition the momentum of the field contained in a certain volume is given by:

$$\mathbf{G} = \int \mathbf{g}\, d\tau = \frac{1}{c}\int \mathbf{S}\, d\tau = \frac{1}{4\pi}\int [\mathbf{EH}]\, d\tau. \tag{16}$$

The assumptions (15) and (16) are suggested by considerations of the energy and momentum balance of a charge distribution in the field. The identity of momentum and energy flow assumed here follows from these considerations. In fact, one can easily prove that the Maxwell-Lorentz equations guarantee—with the assumptions (15) and (16)—the conservation of energy and momentum.

The whole force acting on the charges contained in a certain volume is given by (4) and (5)

$$\mathbf{K} = \int \mathbf{k}\, d\tau = \int \rho\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{H}]\right) d\tau.$$

Since $\mathbf{K}$ is identical with the force of inertia, it also represents the change of the mechanical momentum $\mathbf{u} = m\mathbf{v}c$ of the charges per unit time

$$\mathbf{K} = \frac{d(m\mathbf{v})}{dt} = \frac{1}{c}\frac{d\mathbf{u}}{dt}. \tag{17}$$

Furthermore, the change of the kinetic energy T of the charges is given by

$$\frac{dT}{dt} = \int (\mathbf{k}\mathbf{v})\, d\tau, \tag{18}$$

where $\mathbf{k}$ represents the density of force (4). Inserting ρ and $\rho\mathbf{v}$ from the Maxwell equations (2 c, d) we obtain

$$\frac{1}{c}\frac{d\mathbf{u}}{dt} = \frac{1}{4\pi}\int \left(\mathbf{E}\,\mathrm{div}\,\mathbf{E} - [\mathbf{H}\,\mathrm{curl}\,\mathbf{H}] - \frac{1}{c}[\dot{\mathbf{E}}\mathbf{H}]\right) d\tau \tag{19a}$$

$$\frac{dT}{dt} = \frac{c}{4\pi}\int \mathbf{E}\left(\mathrm{curl}\,\mathbf{H} - \frac{1}{c}\dot{\mathbf{E}}\right) d\tau. \tag{19b}$$

Making use of (2a) the last terms of (19a, b) can also be written

$$-\frac{1}{c}[\dot{\mathbf{E}}\,\mathbf{H}] = -[\mathbf{E}\,\mathrm{curl}\,\mathbf{E}] - \frac{1}{c}\frac{d}{dt}[\mathbf{E}\,\mathbf{H}]$$

$$-(\mathbf{E}\dot{\mathbf{E}}) = -\frac{1}{2}\frac{d}{dt}(E^2 + H^2) - c(\mathbf{H}\,\mathrm{curl}\,\mathbf{E}).$$

According to the definitions (15), (16) the equations (19) become

$$\frac{d\mathbf{u}}{dt} = -\frac{d\mathbf{G}}{dt} + \frac{c}{4\pi}\int (\mathbf{E}\,\mathrm{div}\,\mathbf{E} - [\mathbf{H}\,\mathrm{curl}\,\mathbf{H}] - [\mathbf{E}\,\mathrm{curl}\,\mathbf{E}])\, d\tau \tag{20a}$$

$$\frac{dT}{dt} = -\frac{dU}{dt} + \frac{c}{4\pi}\int (\mathbf{E}\,\mathrm{curl}\,\mathbf{H} - \mathbf{H}\,\mathrm{curl}\,\mathbf{E})\, d\tau. \tag{20b}$$

The integrals on the right-hand side can be transformed into surface

integrals. In (20 b) we simply have according to Gauss's formula

$$\frac{c}{4\pi}\int (\mathbf{E}\,\mathrm{curl}\,\mathbf{H}-\mathbf{H}\,\mathrm{curl}\,\mathbf{E})\,d\tau = -\int \mathrm{div}\,\mathbf{S}\,d\tau = -\oint S_\nu\,d\sigma,$$

where ν denotes the component normal to the surface of the volume considered. For equation (20a) we define a certain tensor which is composed of quadratic terms in the field strengths, the so-called *Maxwell tension tensor*:

$$\begin{aligned} 4\pi T_{xx} &= \tfrac{1}{2}(E_x^2-E_y^2-E_z^2+H_x^2-H_y^2-H_z^2) \\ 4\pi T_{xy} &= 4\pi T_{yx} = E_x E_y + H_x H_y \qquad (21) \\ &\cdots\cdots \end{aligned}$$

The x-component of the vector-divergence of this tensor is just (as one can easily see directly)

$$\begin{aligned} \mathrm{Div}_x\,T &\equiv \frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}+\frac{\partial T_{xz}}{\partial z} \\ &= \frac{1}{4\pi}(E_x\,\mathrm{div}\,\mathbf{E}+H_x\,\mathrm{div}\,\mathbf{H}-[\mathbf{H}\,\mathrm{curl}\,\mathbf{H}]_x-[\mathbf{E}\,\mathrm{curl}\,\mathbf{E}]_x), \end{aligned}$$

where the second term vanishes, by equation (2b).

The integral on the right-hand side in (20a) can be written for the x-component (Gauss's formula also holds for a tensor divergence)

$$\int \mathrm{Div}_x\,T\,d\tau = \oint T_{x\nu}\,d\sigma,$$

where the first index of $T_{x\nu}$ represents the component in the x-direction and the second the component normal to the surface considered.

Thus the equations (20) become

$$\frac{d(u_x+G_x)}{dt} = c\oint T_{x\nu}\,d\sigma \qquad (22a)$$

$$\frac{d(T+U)}{dt} = -\oint S_\nu\,d\sigma. \qquad (22b)$$

In (22) the left-hand side represents the change of energy and momentum of the charged matter and of the field enclosed in a certain volume. On the right-hand side we have an integral of a normal component over the surface of the volume considered. S_ν and $-T_{x\nu}$ have therefore to be interpreted in the following way: S_ν represents the energy of the field passing (in the outward direction) through the surface per unit area and unit time. $-T_{x\nu}c$ represents the x-component of the momentum of the field passing through the surface per unit area and per unit time. The equations (22) give therefore a complete account of the conservation of energy and momentum in the field. The identity of $\mathbf{G}$ (momentum) and $\mathbf{S}/c$

(energy flow) follows necessarily from this energy and momentum balance.

2. Lorentz invariance, momentum, and energy of the field

1. *Lorentz transformations.* The equations of classical electrodynamics are invariant with respect to Lorentz transformations. This fact is, as is well known, the starting-point of the theory of relativity. In this book we shall not discuss the theory of relativity; but since we need some of the transformation relations, we shall summarize them shortly:

A Lorentz transformation is an orthogonal transformation between the 4 space and time coordinates

$$x_1, x_2, x_3, x_4 = ict$$

$$x_i' = \sum_k a_{ik} x_k, \text{ or } x_k = \sum_i a_{ik} x_i'. \tag{1}$$

The a_{ik} form an orthogonal 4-dimensional matrix with the determinant

$$|a_{ik}| = 1. \tag{2}$$

(1) therefore represents in general a rotation in space and a uniform translation. A special transformation between x_1 and x_4 only which signifies a translation along the x-axis with the velocity $v = \beta c$, is given by

$$x_1' = \frac{x_1 + i\beta x_4}{\sqrt{(1-\beta^2)}} = \frac{x_1 - vt}{\sqrt{(1-\beta^2)}}, \qquad x_4' = ict' = \frac{x_4 - i\beta x_1}{\sqrt{(1-\beta^2)}} = \frac{i}{c}\frac{c^2 t - v x_1}{\sqrt{(1-\beta^2)}}. \tag{3}$$

For $v \ll c$, we obtain the Galileo transformation $x_1' = x_1 - vt$ and $t' = t$.

A 4-vector is a set of 4 quantities A_i ($i = 1,\ldots, 4$) which are transformed in the same way as the x_i:

$$A_i' = \sum_k a_{ik} A_k. \tag{4}$$

Similarly, a tensor A_{ik} is defined by the transformation

$$A_{ik}' = \sum_{l,m} a_{il} a_{km} A_{lm}. \tag{5}$$

From (4) and from the orthogonality of the a_{ik} it follows that the scalar product of two vectors is invariant, the scalar product of a vector and a tensor represents a vector, and so on. Thus

$$\sum A_i B_i = \text{inv.}, \qquad \sum_k A_{ik} B_k = C_i. \tag{6}$$

Therefore, the length of a vector also is invariant:

$$\sum_i A_i^2 = \text{inv.} \tag{6'}$$

If ϕ is an invariant function of $x_1,\ldots,x_4$, we see from (1) that if

$$\frac{\partial \phi}{\partial x_i} = B_i, \quad \text{then} \quad B_i' = \sum_k \frac{\partial \phi}{\partial x_k}\frac{\partial x_k}{\partial x_i'} = \sum_k a_{ik} B_k. \tag{7}$$

Thus B_i represents a 4-vector. The derivatives of a 4-vector form a tensor

$$\frac{\partial A_i}{\partial x_k} = B_{ik} \tag{8}$$

therefore, the symbol $\partial/\partial x_i$ can be considered as the i-component of a 4-vector. The symbol

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \sum_i \frac{\partial^2}{\partial x_i^2} \tag{9}$$

is invariant. From (6) it follows that if we put $A_i = \partial/\partial x_i$, then the 4-dimensional divergence of a vector is an invariant:

$$\sum_i \frac{\partial A_i}{\partial x_i} = \text{inv.} \tag{10}$$

The 4-dimensional volume element

$$dx_1\, dx_2\, dx_3\, dx_4 = \text{inv.} \tag{11}$$

is, according to (2), also invariant, whereas with the 3-dimensional volume element this is not the case.

According to (6'), the length of the infinitesimal vector dx_i which may represent the displacement of a particle is an invariant

$$\sum_i dx_i^2 = -d\tau^2.$$

If we divide this equation by dt^2 we obtain

$$\left(\frac{d\tau}{dt}\right)^2 = -\sum_{i=1}^{3}\left(\frac{dx_i}{dt}\right)^2 + c^2 = c^2 - v^2 = c^2(1-\beta^2).$$

Therefore we can define an invariant time element $d\tau$

$$d\tau = c\, dt \sqrt{(1-\beta^2)}, \tag{12}$$

which we call the 'proper time'. The derivative of a vector with respect to τ is again a vector, etc.

2. *Invariance of the Maxwell equations.* The Maxwell-Lorentz equations are shown to be invariant with respect to Lorentz transformations if we succeed in writing them as relations between

4-vectors and tensors. A clue for the 4-dimensional interpretation of these equations can be obtained from the following consideration:

From experience it follows that the *electric charge* is an *invariant* quantity. A charge element enclosed in a given volume $dx_1\,dx_2\,dx_3$ is given by

$$de = \rho\,dx_1\,dx_2\,dx_3 = \text{inv.} \tag{13}$$

Since the 4-dimensional volume element (11) is an invariant, the charge density ρ in (13) must have the transformation properties of the fourth component of a 4-vector. We put, therefore,

$$ic\rho = i_4.$$

Furthermore, the x-component of the current density is given by

$$i_x = \rho v_x = \rho\frac{dx_1}{dt} = i_4\frac{dx_1}{dx_4}.$$

Since i_4 is transformed in the same way as dx_4, i_x is transformed as dx_1 and represents, therefore, the first component of a 4-vector. Therefore charge and current density form together a 4-vector:

$$\rho v_x = i_1, \qquad ic\rho = i_4. \tag{14}$$

We consider now the equations § 1 (11) for the potentials. Both equations have on the left-hand side the operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$, which according to (9) is invariant. The right-hand sides of the two equations (11) represent just the first three and the fourth components respectively of the charge-current vector (14). Therefore, the scalar and vector potential together must also represent a 4-vector

$$A_x = A_1, \qquad A_4 = i\phi. \tag{15}$$

The condition $\operatorname{div}\mathbf{A} + \dot{\phi}/c = 0$, which we have imposed on the potentials is also Lorentz invariant. It can be written as a 4-dimensional divergence:

$$\sum_i \frac{\partial A_i}{\partial x_i} = 0. \tag{16}$$

The field strengths are obtained from the potentials by differentiation. If we use the 4-dimensional notation (15) we have from § 1 eq. (7 a) and (7 b)

$$\left.\begin{aligned} iE_x &= -\frac{i}{c}\dot{A}_x - i\frac{\partial\phi}{\partial x} = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1} \\ H_x &= \operatorname{curl}_x\mathbf{A} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}. \end{aligned}\right\} \tag{17}$$

According to (8), $\partial A_s/\partial x_r = \bar{f}_{rs}$ represents a tensor.

As one can see immediately from the transformation formulae the difference

$$\frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i} = \bar{f}_{ki} - \bar{f}_{ik} = f_{ki} = -f_{ik}$$

has also the character of a tensor. It is *antisymmetrical*. Thus the field strengths **E** and **H** together form an antisymmetrical 4-tensor

$$H_x = f_{23}, \quad H_y = f_{31}, \quad H_z = f_{12}; \quad iE_x = f_{41}, \ldots . \tag{18}$$

The general transformation formulae (3) give us immediately the rules according to which the field strengths have to be transformed in a moving system of coordinates. Considering the case of a uniform translation along the x-axis, we obtain, according to (3), (5), and (18):

$$\left.\begin{aligned} E'_x &= E_x & H'_x &= H_x \\ E'_y &= (E_y - \beta H_z)\gamma & H'_y &= (H_y + \beta E_z)\gamma \\ E'_z &= (E_z + \beta H_y)\gamma & H'_z &= (H_z - \beta E_y)\gamma \\ & & \gamma &= 1/\sqrt{(1-\beta^2)}. \end{aligned}\right\} \tag{19}$$

In the formula (19) it has to be understood that **E**, **H** are functions of the 4 coordinates x_i, whereas **E′**, **H′** are considered as functions of the transformed coordinates x'_i. The values of x'_i which have to be inserted as arguments on the left-hand side of (19) are those which are obtained from x_i on the right-hand side by the transformation (3).

We can now easily write down the Maxwell-Lorentz equations in vector and tensor form. We consider first the two inhomogeneous equations § 1 (2 c) and (2 d):

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \qquad \operatorname{curl} \mathbf{H} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}\rho\mathbf{v}.$$

Using the 4-dimensional notation (14) and (18) these two equations may be written in the *unified* form

$$\sum_l \frac{\partial f_{kl}}{\partial x_l} = \frac{4\pi}{c} i_k \qquad (k = 1, \ldots, 4). \tag{20a}$$

The right-hand side represents the 4-vector of the charge and current density. The left-hand side is the 4-dimensional divergence of a tensor and is therefore also a 4-vector. Equation (20 a) is thus a relation between two vectors.

The second homogeneous pair of Maxwell's equations § 1 (2 a, b) is

$$\operatorname{div} \mathbf{H} = 0 \qquad \operatorname{curl} \mathbf{E} + \frac{1}{c}\dot{\mathbf{H}} = 0.$$

In the 4-dimensional notation (18) they become respectively

$$\left.\begin{aligned} \frac{\partial f_{23}}{\partial x_1}+\frac{\partial f_{31}}{\partial x_2}+\frac{\partial f_{12}}{\partial x_3} &= 0 \\ \frac{\partial f_{43}}{\partial x_2}+\frac{\partial f_{24}}{\partial x_3}+\frac{\partial f_{32}}{\partial x_4} &= 0. \end{aligned}\right\} \tag{21}$$

$\partial f_{ik}/\partial x_l = t_{ikl}$ forms a tensor of the third rank. Applying the transformation formulae (1) one can easily see that the same is also true for the sum $t_{ikl}+t_{kli}+t_{lik}$. Therefore, both equations (21) can be written as a single tensor equation

$$t_{ikl}+t_{kli}+t_{lik} = 0, \qquad t_{ikl} = \frac{\partial f_{ik}}{\partial x_l} \tag{20b}$$

$$(i, k, l = 1,\ldots, 4).$$

Thus we have proved the Lorentz invariance of the Maxwell-Lorentz field equations.

3. *The Lorentz force. Momentum and energy of a particle.* To show the invariance of the whole system of equations of classical electrodynamics, we have finally to write the formula for the Lorentz force in 4-dimensional notation. The formula in question is (§ 1 (4))

$$c\mathbf{k} = c\rho\mathbf{E}+\rho[\mathbf{vH}], \tag{22}$$

where $\mathbf{k}$ is the force density. The right-hand side is simply the scalar product of a vector and a tensor, namely $\sum_k f_{ik} i_k$ $(i = 1, 2, 3)$. Therefore, the force density must represent the spatial part of a 4-vector, and we can write (22):

$$k_i = \frac{1}{c}\sum_k f_{ik} i_k. \tag{23}$$

Only the first 3 components of (23) are identical with (22). The fourth component simply defines k_4: we obtain

$$k_4 = \sum_k f_{4k} i_k = \frac{i}{c}\rho(\mathbf{Ev}). \tag{24}$$

k_4 therefore represents the work which the field performs on the charge per unit volume and per unit time.

If we insert for the charge-current vector i_k the value (20 a) taken from the Maxwell equation, the equation (23) becomes:

$$k_i = \frac{1}{4\pi}\sum_{k,l} f_{ik}\frac{\partial f_{kl}}{\partial x_l}. \tag{25}$$

The right-hand side can be written also as the 4-dimensional divergence of a tensor. For this purpose we define the symmetrical tensor:

$$T_{ik} = \frac{1}{4\pi}\Big[\sum_l f_{il} f_{lk} + \tfrac{1}{4}\delta_{ik}\sum_{l,m} f_{lm}^2\Big], \tag{26}$$

where

$$\delta_{ik} = \begin{cases} 0 & (i \neq k) \\ 1 & (i = k). \end{cases}$$

(δ_{ik} represents a tensor because of the orthogonality of the Lorentz transformation.) The right-hand side of (25) becomes then simply (according to (20 b)):

$$k_i = \sum_k \frac{\partial T_{ik}}{\partial x_k}. \tag{27}$$

The physical significance of this tensor becomes clear if we write it in the 3-dimensional notation. Substituting the f_{ik} from (18) we obtain the following scheme:

$$T_{ik} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} & -i/cS_x \\ T_{yx} & T_{yy} & T_{yz} & -i/cS_y \\ T_{zx} & T_{zy} & T_{zz} & -i/cS_z \\ -i/cS_x & -i/cS_y & -i/cS_z & u \end{pmatrix} \tag{28}$$

The space-time part of this tensor represents the energy flow **S**, the pure time part the energy density u. The pure space part is identical with the Maxwell tension tensor, introduced in § 1 (21) in the equation for the momentum balance of the field, and represents the flow of momentum. All these quantities together form, therefore, a 4-dimensional tensor, which we may call the generalized tension tensor.

Equation (27) is identical with the equations § 1 (22) which express the energy and momentum balance. To show this, we have only to write down the space and time derivatives separately, for instance

$$k_x = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} + \frac{1}{ic}\frac{\partial T_{x4}}{\partial t} = \mathrm{Div}_x\, T - \frac{1}{c}\frac{\partial g_x}{\partial t}.$$

Integrating over all space we obtain

$$\int k_x\, d\tau = -\frac{1}{c}\frac{\partial G_x}{\partial t} + \oint T_{x\nu}\, d\sigma = K_x, \tag{29a}$$

which is identical with § 1 (22 a). In the same way the fourth component k_4 of (27) gives, according to (24),

$$\frac{c}{i}\int k_4\, d\tau = \int \rho(\mathbf{Ev})\, d\tau = -\oint S_\nu\, d\sigma - \frac{\partial U}{\partial t}, \tag{29b}$$

which is identical with equation § 1 (22 b) for the energy balance.

If we integrate (27) over a 4-dimensional volume—which is an invariant—we obtain another very important 4-vector, namely the vector of the *momentum and energy* of the total charge u_i. If the charge is concentrated in a small volume and if the space integration is extended over the whole of this volume, we may think of the integrated charge as a 'particle' and shall simply speak of the energy and momentum of the particle. The left-hand side of (27) becomes

$$\left.\begin{aligned} c\int k_x\, d\tau dt = c\int K_x\, dt = u_x = u_1 \\ c\int k_4\, d\tau dt = i\int \rho(\mathbf{Ev})\, d\tau dt = iT, \end{aligned}\right\} \tag{30}$$

i.e. the *momentum* $\mathbf{u}$ *and kinetic energy* T (times i) of the *particle.* They form therefore together a 4-vector. The formulae (3) and (30) give us then the transformation formulae for the kinetic energy and momentum of a particle, which may be written as follows:

$$\left.\begin{aligned} u_x' &= (u_x - \beta T)\gamma, \qquad u_y' = u_y \\ T' &= (T - \beta u_x)\gamma, \qquad \gamma = 1/\sqrt{(1-\beta^2)}. \end{aligned}\right\} \tag{31}$$

The length of the vector u_i is invariant:

$$-\sum_i u_i^2 = T^2 - (u_x^2 + u_y^2 + u_z^2) = \mu^2. \tag{32}$$

This invariant μ has a simple significance; it represents obviously the *energy of the particle at rest.* ($u_x = u_y = u_z = 0$.)

From (32) we can therefore obtain the momentum and energy as functions of the velocity, if we assume the particle is at rest in one (the dashed) system of coordinates, so that $u_x' = 0$. Then we obtain according to (31) and (32)

$$u_x = \frac{\mu\beta}{\sqrt{(1-\beta^2)}}, \qquad T = \frac{\mu}{\sqrt{(1-\beta^2)}}, \qquad \beta = v/c. \tag{33}$$

These are the famous *Einstein formulae* for the energy and momentum of a moving particle. Since for $v \ll c$ the momentum u_x must be identical with the classical value $mv_x c$, the rest energy μ takes the form

$$\mu = mc^2. \tag{34}$$

For a slow particle ($\beta \ll 1$) the kinetic energy becomes

$$T = \mu + \tfrac{1}{2}\mu\beta^2 = mc^2 + \tfrac{1}{2}mv^2, \tag{33'}$$

i.e. we obtain, besides the rest energy μ, the ordinary kinetic energy $mv^2/2$.

Finally we consider the relativistic *equation of motion of a charged*

particle in a given external field. The total force acting on the particle was given by

$$\mathbf{K} = e\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right).$$

In the non-relativistic theory **K** has to be equated to the time derivative of the momentum. As we shall see immediately, one has in relativistic dynamics to insert for **K** the time derivative of our 4-vector u_i, so that the equation of motion becomes

$$K_x = \frac{1}{c}\frac{du_x}{dt} = \frac{d}{dt}\frac{mv_x}{\sqrt{(1-\beta^2)}} = e\left(E_x + \frac{1}{c}[\mathbf{vH}]_x\right). \tag{35}$$

In fact (35) can be written as a 4-dimensional vector equation. Dividing by $\sqrt{(1-\beta^2)}$ the right-hand side becomes according to (18) and (33)

$$\frac{e}{\mu}\sum_k f_{1k}u_k.$$

The left-hand side represents the derivative with respect to the proper time τ (12) which is invariant. Thus (35) can be written as

$$\frac{du_i}{d\tau} = \frac{e}{\mu}\sum_k f_{ik}u_k. \tag{36}$$

(36) represents the relativistic equation of motion of a particle in an external field.

The fourth component of the 4-vector u_i represents the *kinetic* energy T. For a particle moving in an external field one can also consider the *total energy* E, i.e. kinetic energy+potential energy. The potential energy is simply equal to the scalar potential ϕ multiplied by e.

Since ϕ is the fourth component of a 4-vector the total energy of the particle can also be related to a 4-vector p_i

$$\left.\begin{aligned} p_i &= u_i + eA_i, \\ p_1 = u_x + eA_x, \quad \ldots, \quad p_4 &= iE = i(T + e\phi). \end{aligned}\right\} \tag{37}$$

Thus, in an external field with a vector potential **A**, the space components of our 4-vector u_i do not represent the total momentum of the particle but a quantity which we may call the *kinetic momentum*. This quantity bears the same relation to the ordinary momentum as the 'kinetic energy' does to the total energy. In the literature $u_x = p_x - eA_x$ (divided by mc) is usually called *four-velocity*.

The space components of p_i representing the total momentum

differ from the 'kinetic momentum' u_i by the vector potential, which is, however, in most cases very small. If **A** is zero, the kinetic momentum **u** is identical with the total momentum **p**.

According to (32) the momentum energy vector p_i (37) satisfies the important equation

$$\sum_i u_i^2 = \sum_i (p_i - eA_i)^2 = -\mu^2. \tag{38}$$

4. *Non-electromagnetic nature of the inertial mass.* The right-hand side of equation (27) integrated over a 4-dimensional volume represents also a 4-vector, viz.

$$\int \sum_k \frac{\partial T_{ik}}{\partial x_k} dx_1 \dots dx_4. \tag{39}$$

At first sight one would perhaps be inclined to call this 4-vector the energy-momentum vector of the field. But to do so would in general have no significance. The expression (39) consists of two parts. Separating the time and space derivatives we obtain from (28) for the x- and t-components the time integrals of (29)

$$c \int \sum_k \frac{\partial T_{xk}}{\partial x_k} dx_1 dx_2 dx_3 dt = -G_x + c \int dt \oint T_{xv} d\sigma \tag{40a}$$

$$ic \int \sum_k \frac{\partial T_{4k}}{\partial x_k} dx_1 dx_2 dx_3 dt = +U + \int dt \oint S_v d\sigma. \tag{40b}$$

In § 1 we have assumed that the first terms, U and **G**, represent energy and momentum of the field, whereas the second terms represent the time integrals of the flow of energy and momentum through the surface. The latter integrals depend, for a given instant of time t, not only upon the field at this time, but also upon the field at all previous times (they are time integrals). But a quantity which depends upon the past history of the system cannot of course be called the energy or momentum of the field. On the other hand, the quantities U and G_x, which for good reasons we have interpreted as energy and momentum of the field, do *not* form a 4-vector. (According to (28) they are parts of a 4-tensor.) They behave quite differently from the corresponding quantities for a particle. Thus, *in general the field has not the properties of a particle with regard to the Lorentz invariance of energy and momentum.*

This is the reason for the lack of success of some older theories

which tried to explain the inertial mass of an electron by its pure electromagnetic properties. The idea of these theories, which are chiefly due to Abraham, was this:† the electron is assumed to have no *mechanical* inertial mass at all. But it produces a field which has a certain amount of energy and momentum, and therefore also a certain inertia (since the field is changed if the velocity of the electron is changed, see § 4). The inertia, which we observe as mass if we accelerate an electron in an external field, should be entirely due to this inertia of its field. One has then to consider the external field and the field produced by the electron separately. In the equations § 1 (22) for the energy and momentum balance, for instance, it is only necessary for $\mathbf{G}$ and U to insert the external field, whereas $\mathbf{u}$ and T represent the energy and momentum of the self-produced field of the electron. From the relativistic point of view it becomes clear that this idea cannot be correct. It has been shown by many experiments that $\mathbf{u}$ and T for an electron actually behave like a 4-vector (velocity dependence of the inertial mass of the electron). But the self-produced field of an electron has—as we have seen—quite a different transformation character and depends upon the velocity in a different way. Therefore, we must attribute to the electron a *special inertial* mass of non-electromagnetic nature, and we must insert for $\mathbf{G}$ and U the whole field, including the field produced by the electron. This, however, gives rise to some other very grave difficulties with regard to the reaction of the field on the electron. We shall return to these questions in § 4 from another point of view. (Compare also § 25.)

5. *'Particle Properties' of light waves.* There are, however, cases where the total energy and momentum of a field themselves form a 4-vector. This is true, for instance, for a light wave of any shape and finite extension. We shall prove in general: *If the field differs from zero only within a certain given volume V, and if no charges are present inside this volume, then the total energy and momentum of the field form a 4-vector.*‡ Since the density of force then also vanishes, we have according to (27)

$$\sum_k \frac{\partial T_{ik}}{\partial x_k} = 0, \qquad T_{ik} = 0 \text{ on the boundary.} \tag{41}$$

† M. Abraham, *Theorie der Electrizität*, II, 5th edition, Leipzig 1923, and 6th edition by R. Becker, Leipzig 1933. ‡ Cf. for instance, Abraham-Becker, p. 308.

To prove the above statement we shall first consider an arbitrary vector A_i which also satisfies these conditions (41), namely

$$\sum_i \frac{\partial A_i}{\partial x_i} = 0$$

and $A_i = 0$ on the boundary. From Gauss's formula applied to the 4-dimensional divergence of a 4-vector, it follows that the surface integral of the normal component of A_i over the surface of a 4-dimensional volume vanishes. For this 4-dimensional volume we chose a cylinder which is parallel to the x_4-axis. The base is the 3-dimensional volume considered. The cylinder is bounded by a section $x_4 = \text{const.}$ and by another section $x_4' = \text{const.}$, where x_4' denotes the time referred to a *moving* system of coordinates. (See Fig. 1, the x_2 and x_3 coordinates are omitted.)

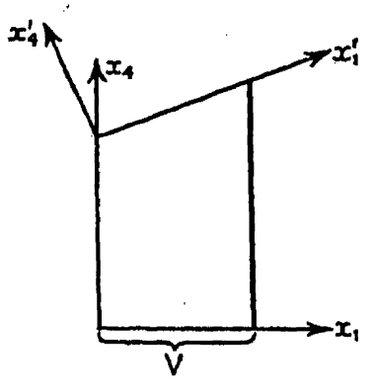

FIG. 1. A 4-dimensional integration.

The normal components of A_i are $-A_4$ and $+A_4'$ respectively on these two sections and zero on the walls of the cylinder. Thus Gauss's formula gives

$$-\int A_4\, d\tau + \int A_4'\, d\tau' = 0. \qquad (42)$$

Alternatively, the integral of A_4 over the volume V is, subject to the suppositions made above, according to (42) an invariant. Putting now

$$\sum_k b_k T_{ik} = A_i,$$

where b_k is an arbitrary constant vector, we obtain from the invariance of $\int A_4\, d\tau$

$$\int T_{4k}\, d\tau = k\text{-component of 4-vector.} \qquad (43)$$

The conditions (41) for which (43) holds are satisfied for any light wave with a finite extension which may have been emitted from a light source some time ago. In this case the *momentum* **G** *and energy* U of the field form a *4-vector* G_i, which behaves as regards its transformation properties exactly like the *energy and momentum* of a particle. In particular, the transformation formulae for a moving system of coordinates are

$$G_x' = (G_x - \beta U)\gamma, \qquad G_y' = G_y, \qquad U' = (U - \beta G_x)\gamma$$
$$1/\gamma = \sqrt{(1-\beta^2)}, \qquad \beta = v/c. \qquad (44)$$

In the case of a plane wave, $E_y = a \sin \nu(x/c - t) = H_z$, the length of

the vector G_i is zero. Since the momentum and energy contained in a volume V are given by

$$G_x = \frac{V}{8\pi}a^2, \qquad U = \frac{V}{8\pi}a^2,$$

we have

$$\sum G_i^2 = G_x^2 - U^2 = 0. \tag{45}$$

This can also be expressed as follows: the rest energy of a plane wave is zero. A plane wave cannot be transformed to rest.

Equation (45) does not, however, hold in the general case. A spherical light wave emitted from a point source (the source may be removed after the emission) has a momentum zero, but a finite energy.

We shall see later in § **6** that every field can be composed of two parts: one part which is given by superposing light waves and for which therefore energy and momentum form a 4-vector, and another part containing the static field for which this is not the case. In the quantum theory only the first part is subjected to a quantization, giving rise to the existence of light quanta which behave also in some other ways like particles; the second, static part of the field remains, on the other hand, unquantized.

3. Field of a point charge and emission of light

The general Maxwell-Lorentz theory of § **1** gives immediately the field which is produced by any given charge distribution. Since we shall see that in some cases the field at a large distance from the charges consists of waves changing periodically with the time, this theory includes also a theory of the emission of light. For the applications to atomic physics the most important case is where the charge producing the field is a point-charge. We shall therefore first work out the field produced by a *point-charge* moving in an arbitrary way (we need this for several purposes). We shall then apply the formulae to a simple model of a light source.

1. *The Wiechert potentials.* The field of a charge distribution ρ is in general given by the equations § **1** (13) for the retarded potentials:

$$\phi(P) = \int \frac{\rho(P', t')}{r_{PP'}}\, d\tau' \qquad (t' = t - r_{PP'}/c), \tag{1a}$$

$$\mathbf{A}(P) = \frac{1}{c}\int \frac{\rho \mathbf{v}(P', t')}{r_{PP'}}\, d\tau', \tag{1b}$$

where t' represents the retarded time of the point P'. If the charges are in motion, we must be careful in carrying out the transition to a point-charge. We may not, for instance, write simply

$$\left.\frac{e}{r}\right|_{t-r/c}$$

for the integral (1 a). Since we have to insert in (1) for each point P' another retarded time, the integral $\int \rho(P', t')\, d\tau'$ would not represent the total charge. Before carrying out the transition to a point-charge we must transform the integrals (1) into integrals over the charge elements de. This can be done in the following way: we assume that at a given time t' all charge elements have the same velocity $\mathbf{v}(t')$. We then consider a spherical shell of thickness dr at a distance r from P. A volume element of this shell will be $d\sigma dr = d\tau$. The contribution to the integral (1 a) will be the density ρ (divided by r) which a light wave meets when it is contracting with the velocity c and passing the outer surface of this shell at a time t'. During the time $dt = dr/c$ which this light wave would take to pass the shell dr, a certain amount of charge will stream through the inner surface of the shell. This amount of charge (per unit area) is given by

$$-\rho\frac{(\mathbf{vr})}{r}\,dt = -\rho\frac{(\mathbf{vr})}{r}\frac{dr}{c}$$

if the direction of $\mathbf{r}$ is chosen to be that of the line PP'. The charge element de collected by the light wave during the time dt is therefore

$$de = \rho\left(1+\frac{(\mathbf{vr})}{rc}\right)d\tau.$$

The integrand of (1) becomes then

$$\rho\, d\tau = \frac{de}{1+(\mathbf{vr})/rc}. \tag{2}$$

Inserting this expression in the integrals (1) we can now carry out the transition to a point-charge and obtain

$$\phi(P,t) = \left.\frac{e}{r+(\mathbf{vr})/c}\right|_{t-r/c}, \tag{3 a}$$

$$\mathbf{A}(P,t) = \left.\frac{1}{c}\frac{e\mathbf{v}}{r+(\mathbf{vr})/c}\right|_{t-r/c} \tag{3 b}$$

All quantities in (3) have to be taken at the retarded time $t-r/c$. Also r is of course the distance at this retarded time t'. The potentials (3), therefore, contain the time t implicitly in a rather complicated way.

The expressions (3) were first obtained by Lienard and Wiechert. They in general represent the potentials of a particle.

2. *Field strengths of an arbitrarily moving point-charge.* The field strengths **E**, **H** can be deduced from (3) by differentiation

$$\mathbf{E} = -\operatorname{grad}\phi - \frac{1}{c}\dot{\mathbf{A}}$$

$$\mathbf{H} = \operatorname{curl}\mathbf{A}.$$

The derivatives have to be taken with respect to the time t and coordinates at P. But since the motion of the particle at the time t' is given by $\mathbf{r}(t')$ and $\mathbf{v}(t') = \partial\mathbf{r}/\partial t'$ the quantities $\mathbf{r}$, $\mathbf{v}$ occurring in (3) are given as functions of the retarded time t'. We must, therefore, first express the derivatives with respect to t in terms of the derivatives with respect to t'. The retarded time t' is defined by the distance r at t'

$$r(t') = c(t - t'). \tag{4}$$

Thus, differentiating with respect to t,

$$\frac{\partial r}{\partial t} = \frac{\partial r}{\partial t'}\frac{\partial t'}{\partial t} = \frac{(\mathbf{rv})}{r}\frac{\partial t'}{\partial t} = c\left(1 - \frac{\partial t'}{\partial t}\right),$$

or

$$\frac{\partial t'}{\partial t} = \frac{1}{1+(\mathbf{vr})/rc} = \frac{r}{s}, \tag{5}$$

where we have used the abbreviation

$$s = r + \frac{(\mathbf{vr})}{c}. \tag{6}$$

Also because of equation (4) t' is a function of the coordinates of P. Therefore we have ($\mathbf{r}$ having the direction from P to P'):

$$\operatorname{grad} t' = -\frac{1}{c}\operatorname{grad} r(t') = -\frac{1}{c}\left(\frac{\partial r}{\partial t'}\operatorname{grad} t' - \frac{\mathbf{r}}{r}\right)$$

or, according to (6),

$$\operatorname{grad} t' = \frac{\mathbf{r}}{cs}. \tag{7}$$

For the derivates of s we obtain according to (5), (7)

$$\left.\begin{aligned} \frac{\partial s}{\partial t} &= \frac{\partial s}{\partial t'}\frac{\partial t'}{\partial t} = \frac{r}{s}\left(\frac{(\mathbf{rv})}{r} + \frac{v^2}{c} + \frac{(\mathbf{r}\dot{\mathbf{v}})}{c}\right) \\ \operatorname{grad} s &= -\frac{\mathbf{r}}{r} - \frac{\mathbf{v}}{c} + \frac{\mathbf{r}}{cs}\left(\frac{(\mathbf{rv})}{r} + \frac{v^2}{c} + \frac{(\mathbf{r}\dot{\mathbf{v}})}{c}\right) \end{aligned}\right\} \tag{8}$$

In these formulae $\dot{\mathbf{v}}$ denotes the derivative of $\mathbf{v}$ with respect to t'. Also $\mathbf{v}$ depends only upon t'. Thus

$$\frac{\partial \mathbf{v}}{\partial t} = \dot{\mathbf{v}}\frac{r}{s}, \qquad \operatorname{curl}\mathbf{v} = -[\dot{\mathbf{v}}\operatorname{grad}t'] = \frac{[\mathbf{r}\dot{\mathbf{v}}]}{cs}. \tag{9}$$

For the field strengths we have, according to (3),

$$\frac{\mathbf{E}}{e} = -\operatorname{grad}\frac{1}{s} - \frac{1}{c^2}\frac{\partial}{\partial t}\frac{\mathbf{v}}{s} = \frac{1}{s^2}\operatorname{grad}s - \frac{1}{c^2 s}\frac{\partial \mathbf{v}}{\partial t} + \frac{\mathbf{v}}{c^2 s^2}\frac{\partial s}{\partial t}$$

$$\frac{\mathbf{H}}{e} = \frac{1}{c}\operatorname{curl}\frac{\mathbf{v}}{s} = \frac{1}{sc}\operatorname{curl}\mathbf{v} + \frac{1}{cs^2}[\mathbf{v}\operatorname{grad}s].$$

Inserting our formulae (8), (9) and (6), we obtain finally

$$\frac{\mathbf{E}}{e} = -\frac{1-\beta^2}{s^3}\left(\mathbf{r}+\frac{\mathbf{v}}{c}r\right) + \frac{1}{s^3c^2}\left[\mathbf{r}\left[\mathbf{r}+\frac{\mathbf{v}}{c}r, \dot{\mathbf{v}}\right]\right] \tag{10a}$$

$$\frac{\mathbf{H}}{e} = \frac{[\mathbf{E}\mathbf{r}]}{r}, \qquad \beta = \frac{v}{c}. \tag{10b}$$

The magnetic field strength is always perpendicular to $\mathbf{E}$ and to $\mathbf{r}$, whereas the electric field strength has, besides a component which is perpendicular to $\mathbf{r}$, a component also in the direction of $\mathbf{r}$.

In these expressions (10), all quantities are of course understood to refer to the retarded time $t' = t - r/c$, where r itself is the distance at the retarded time.

The formulae (10) are general, in as much as they are valid for any given motion of the particle even when the velocity is comparable with the velocity of light. But they hold only as long as the particle can be considered as a point-charge. They do not hold at distances which are comparable with the electronic radius (see § 4).

The field (10 a) is composed of two parts which behave quite differently. The first part depends only upon the velocity and decreases with r^{-2} for long distances from the particle. It represents the static (Coulomb) part of the field and reduces simply to $-\mathbf{r}/r^3$ for $\mathbf{v} = 0$. For $\mathbf{v} \neq 0$ this part can readily be deduced from the Coulomb field by means of a Lorentz transformation (§ 2 eq. (19)). The second part is proportional to the *acceleration* $\dot{\mathbf{v}}$, and decreases only as r^{-1} for long distances from the particle. This part of the field is *transverse*, the field strengths being perpendicular to the radius vector $\mathbf{r}$. We shall see that it gives rise to emission of light. We call the region where the second part preponderates the '*wave zone*'. The

distinction between these two parts of the field has, however, only a meaning if the region which the electron passes during the time r/c is small compared with r itself (quasi-periodic motion, or if $v/c \ll 1$).

3. *The Hertzian vector of a system of charges. Dipole and Quadripole momentum.* As a first application of our general theory, we shall consider the emission of light by a system of point-charges. We assume that a certain number of particles with charges e_k are all situated in the neighbourhood of a centre Q. The distance of Q from the point P at which we consider the field may be denoted by $\mathbf{R}$ (direction $P \to Q$) and the distance $P \to e_k$ by $\mathbf{r}_k$. Each charge can then be described by a vector representing the displacement of e_k from the centre Q

$$\mathbf{x}_k = \mathbf{r}_k - \mathbf{R}. \tag{11}$$

We assume that all displacements are small compared with R, and furthermore that the velocities of all particles are small compared with the velocity of light.

$$|x_k| \ll R, \qquad v_k \ll c. \tag{12}$$

The field is of course the result of the superposition of the fields arising from all single charges.

In the expressions (10) one has, however, to insert for each particle another retarded time $t'_k = t - r_k/c$. But since all displacements are small, it will be convenient to introduce a new time T, the retarded time of the centre Q, viz.

$$T = t - \frac{R}{c}, \tag{13}$$

and to express the field strengths as functions of this time T.

In the first approximation we may replace in (10) all derivates with respect to t'_k by the derivative with respect to T, which is the same for all particles. Thus

$$\dot{\mathbf{v}}_k = \mathbf{x}''_k = \frac{\partial^2 \mathbf{x}_k}{\partial T^2},$$

where we have denoted the derivative with respect to T by a dash. Furthermore we may replace $\mathbf{r}_k$ by R and neglect all terms containing v/c. Then s also becomes equal to R.

Introducing the sum of all displacements

$$\mathbf{Z} = \sum_k e_k \mathbf{x}_k(T), \tag{14}$$

we obtain for this part of the field which is proportional to the acceleration, and which decreases as R^{-1} (wave zone)

$$\left.\begin{aligned}\mathbf{E}(t) &= \frac{1}{R^3c^2}[\mathbf{R}[\mathbf{R}\mathbf{Z}'']]_{t-R/c}\\ \mathbf{H}(t) &= \frac{1}{R^2c^2}[\mathbf{R}\mathbf{Z}'']_{t-R/c}.\end{aligned}\right\} \tag{15}$$

(15) has to be taken at the retarded time $t-R/c = T$ of the centre.

Before we discuss these simple formulae we shall consider the next approximation. In (15) we have equated the retarded times for all particles to the retarded time of their centre. We shall now take into account, as the next approximation, the fact that these times are not identical, but that a certain time is necessary for the light to pass from one particle to another. This effect of the retardation, as we shall see, will give the quadripole radiation. For this purpose we have to evaluate the field strengths up to terms of the first power of v/c and to express them as functions of the common time T. The displacements x_k, however, will be put equal to zero in the final expressions for the field strengths (i.e. we calculate the field in the wave zone only).

Developing the electric field strength (10a) to the first power of v/c we obtain, according to (6),

$$\mathbf{E} = \sum_k e_k[\mathbf{R}[\mathbf{R}\mathbf{q}_k]]\frac{1}{R^3c^2}, \qquad \mathbf{q}_k = \dot{\mathbf{v}}_k - 2\frac{(\mathbf{R}\mathbf{v}_k)}{Rc}\dot{\mathbf{v}}_k - \mathbf{v}_k\frac{(\mathbf{R}\dot{\mathbf{v}}_k)}{Rc}. \tag{16}$$

Here we have replaced $\mathbf{r}_k$ by $\mathbf{R}$, neglecting terms containing the displacements. They would not give any contribution to the field in the wave zone. In (16) $\mathbf{v}_k$ and $\dot{\mathbf{v}}_k$ still have to be understood as derivatives with respect to the time t'_k. To express them as derivatives with respect to T we write (for each k)

$$\left.\begin{aligned}\mathbf{v} &= \frac{\partial\mathbf{x}(t')}{\partial t'} = \frac{\partial\mathbf{x}(t')}{\partial T}\frac{\partial T}{\partial t'} = \mathbf{x}'\frac{\partial T}{\partial t'}\\ \dot{\mathbf{v}} &= \frac{\partial\mathbf{v}}{\partial T}\frac{\partial T}{\partial t'} = \mathbf{x}''\left(\frac{\partial T}{\partial t'}\right)^2 + \mathbf{x}'\left(\frac{\partial}{\partial T}\frac{\partial T}{\partial t'}\right)\frac{\partial T}{\partial t'}.\end{aligned}\right\} \tag{17}$$

In these formulae $\mathbf{x}'$ denotes the derivative of $\mathbf{x}$ (which is still given as a function of t') with respect to T. This of course only makes sense if we later express $\mathbf{x}$ as a function of T. $\partial T/\partial t'$ can be taken from the definition of T, viz.

$$c(t'-T) = R - r(t'),$$

so that

$$\frac{\partial T}{\partial t'} = 1 + \frac{(\mathbf{v}\mathbf{R})}{Rc} = 1 + \frac{(\mathbf{x}'\mathbf{R})}{Rc}, \tag{18}$$

where we have again replaced $\mathbf{r}$ by $\mathbf{R}$ and $\mathbf{v}$ by its first approximation $\mathbf{x}'$, since it occurs in (18) only in the second-order term. Inserting (18) in (17) we find for $\dot{\mathbf{v}}$:

$$\dot{\mathbf{v}} = \mathbf{x}'' + 2\mathbf{x}'' \frac{(\mathbf{x}'\mathbf{R})}{Rc} + \mathbf{x}' \frac{(\mathbf{x}''\mathbf{R})}{Rc}. \tag{19}$$

In (16) we can in the second and third term of $\mathbf{q}$ also replace $\mathbf{v}$ and $\dot{\mathbf{v}}$ by their first approximations. $\mathbf{q}_k$ then becomes according to (16) and (19) just

$$\mathbf{q}_k = \mathbf{x}_k''.$$

Introducing again the vector giving the sum of all displacements (*Hertzian vector*)

$$\mathbf{Z}(T) = \sum_k e_k \mathbf{x}_k(t_k'), \tag{20}$$

we obtain for the field strengths in the wave zone, omitting the static part,

$$\left.\begin{aligned} \mathbf{E} &= \frac{1}{R^3c^2}[\mathbf{R}[\mathbf{R}\mathbf{Z}'']]_{t-R/c}, \\ \mathbf{H} &= \frac{1}{R^2c^2}[\mathbf{R}\mathbf{Z}'']_{t-R/c}. \end{aligned}\right\} \tag{21}$$

Formally, (21) is identical with (15) giving the field strengths in the first approximation. The difference lies in the definition of $\mathbf{Z}$. According to (20), $\mathbf{Z}$ is considered as a function of the common time T but equated to the sum of the *exact* displacements, still expressed in terms of t_k'.

We have now to express $\mathbf{x}_k(t_k')$ as a function of T. Since t_k' and T differ only slightly, we can develop

$$\mathbf{x}_k(t_k') = \mathbf{x}_k(T) + \mathbf{x}_k' \frac{(\mathbf{x}_k\mathbf{R})}{Rc} + \dots$$

and

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2$$

$$\mathbf{Z}_1 = \sum_k e_k \mathbf{x}_k(T), \qquad \mathbf{Z}_2 = \sum_k e_k \mathbf{x}_k'(T) \frac{(\mathbf{x}_k\mathbf{R})}{Rc}. \tag{22}$$

The first term $\mathbf{Z}_1$ is identical with (14). It represents the *dipole moment* of the charge system. The second term is due to the effect of retardation, since $(\mathbf{x}_k\mathbf{R})/Rc$ represents just the difference of the retarded times for the kth particle and for the centre. It is usually called the *quadripole moment* of the charge system.† It plays, in general, an important role only if the arrangement of the charges is so symmetrical that the dipole moment vanishes.

† This is not quite correct since the term considered contains also a part which gives rise to a magnetic dipole-radiation, which is, however, also of the second order. (See Brinkmann, *Zur Quantenmechanik der Multipolstrahlung, Proefschrift*, Utrecht, 1932.)

4. *Emission of light.* The field given by the equation (21) decreases with the first power of R. Any quadratic expression in the field strengths, for instance, the Poynting vector of the energy-flow, will decrease as R^{-2} and the integral over a spherical surface will therefore be finite and independent of R. Thus the field given by equation (21) gives rise to a finite energy flow through any distant sphere, i.e. to an *emission of radiation.* The radiation is *transverse,* both field strengths being perpendicular to **R** and to each other. The energy flow per unit time through the area $R^2 d\Omega$ becomes

$$\mathbf{S}R^2 d\Omega = \frac{c}{4\pi}[\mathbf{EH}]R^2 d\Omega = \frac{d\Omega}{4\pi c^3}\frac{\mathbf{R}}{R}Z''^2 \sin^2\theta, \tag{23}$$

where θ represents the angle between the direction of the Hertzian vector $\mathbf{Z}''$ and the direction of observation. The energy flow (23) is normal to a spherical surface and proportional to the square of the second time derivative of the Hertzian vector. It has a maximum at right angles to $\mathbf{Z}''$ and vanishes in the direction of $\mathbf{Z}''$. The direction of polarization of this radiation (direction of **E**) is the projection of $\mathbf{Z}''$ on the plane perpendicular to **R**.

The total energy radiated per unit time S is given by integrating (23) over all angles

$$S = \frac{2}{3}\frac{Z''^2}{c^3}. \tag{24}$$

The simplest model of a light source is a harmonic oscillator, i.e. a single charge bound elastically to a centre of force and moving with simple harmonic motion of frequency ν along the x-axis. In this case we can put

$$\mathbf{Z} = e\mathbf{x} = e\mathbf{x}_0 \cos \nu t, \qquad \mathbf{Z}'' = -e\nu^2\mathbf{x}.$$

The radiation emitted from this oscillator is then monochromatic with the same frequency ν. The time average value of (24) becomes

$$S = \frac{2}{3}\frac{e^2}{c^3}\nu^4\overline{x^2} = \frac{1}{3}\frac{e^2}{c^3}\nu^4 x_0^2. \tag{25}$$

The energy emitted per unit time is therefore proportional to ν^4.

4. Reaction of the field, line breadth

In § 3 we have seen that a moving point-charge in general emits radiation. The energy radiated per unit time for a slowly moving particle is given by § 3 eq. (24). According to the equations for the energy balance of the field ((22) § 1), this energy has to be contributed by the forces keeping the charge in motion. As a consequence of the

energy loss due to the radiation, therefore, the kinetic energy of the particle must decrease with the time. In § **3** we have calculated the radiation for the case when the motion of the particle is given. It would not, therefore, be correct to determine the particle's motion only from the external forces acting on it (e.g. the quasi-elastic forces of an oscillator), since the motion is also influenced by the radiation emitted. In order to give a correct account of the conservation of energy we have to consider the *reaction* of the field produced by the charge on its own motion.

We shall find, however, that in general the reaction force of the radiation is small compared with other forces. This makes it possible to consider the reaction effect as a small correction, so that in a first approximation we may assume the motion of the particle to be determined by the external forces only.

We shall calculate this reaction in two different ways. The first way is purely phenomenological: we assume that in the first approximation the motion of the particle is not altered by the radiation emitted and that the latter is given by § **3** eq. (24). We then add as a second approximation in the equation of motion another small term (damping-force) which is determined so that it just preserves the energy balance.

The second method is more profound and, from the point of view of the general theory, more consistent. We shall calculate directly the force which the field produced by the charge exerts on the charge itself. The expression for the reaction force obtained in this way is the same as that obtained from the energy balance. However, this method leads to further results which are principally important in the discussion of the whole radiation theory.

1. *First way: the energy balance.* According to § **3** (24) and (20) the energy radiated per unit time by an accelerated particle moving with small velocity is given by

$$S = \tfrac{2}{3}e^2\dot{\mathbf{v}}^2/c^3. \tag{1}$$

In the first approximation the equation of motion of the particle is

$$\mathbf{K} = m\dot{\mathbf{v}}, \tag{2}$$

where $\mathbf{K}$ represents the external force (m = mass of the electron). To this equation (2) we add another term $\mathbf{K}_s$ ('self-force') which is to take account of the energy loss (1):

$$\mathbf{K}+\mathbf{K}_s = m\dot{\mathbf{v}}. \tag{3}$$

In order to simplify the energy balance § 1 eq. (22) we shall assume that for two certain times t_1, t_2 the state of motion of the particle is the same at t_2 as at t_1. Then also the energy of the field U is the same at t_2 as at t_1. The work performed by the force $\mathbf{K}_s$ during this time interval must then be equal to the total energy radiated

$$\int_{t_1}^{t_2} (\mathbf{K}_s \mathbf{v})\, dt = -\frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2} \dot{\mathbf{v}}^2\, dt. \tag{4}$$

This equation for $\mathbf{K}_s$ can be solved in general.

Integrating (4) by parts we obtain

$$\int_{t_1}^{t_2} (\mathbf{K}_s \mathbf{v})\, dt = +\frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2} (\mathbf{v}\ddot{\mathbf{v}})\, dt. \tag{5}$$

(5) is satisfied if we put for each instant of time:

$$\mathbf{K}_s = \frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{v}}. \tag{6}$$

(6) gives the desired reaction force. It is proportional to the *time derivative of the acceleration*. For a particle moving with simple harmonic motion one can of course replace $\ddot{\mathbf{v}}$ by $-\nu^2\mathbf{v}$.

It can readily be proved that the force (6) also gives a correct account of the conservation of momentum and angular momentum.

This deduction is only correct so long as the reaction force (6) is small compared with the other forces. For a harmonic oscillator this is always the case as long as the frequency is not too high. Putting $\mathbf{x} = \mathbf{x}_0 \cos \nu_0 t$ the quasi-elastic force is $\nu_0^2 m\mathbf{x}$ and our condition gives:

$$m\nu_0^2 \gg \frac{e^2}{c^3}\nu_0^3, \tag{7}$$

or introducing the wave-length λ

$$\lambda = \frac{c}{\nu_0} \gg \frac{e^2}{mc^2} \equiv r_0. \tag{7'}$$

r_0 represents, as we shall see in subsection 3, the classical electronic radius and is of the order of 10^{-13} cm. Thus (7) holds for all wave-lengths large compared with r_0. In the next subsection we shall discuss this condition from a more general point of view.

(7) is satisfied for all kinds of radiation known at present (including γ-rays) except for the high-energy cosmic radiation. (See Chapter V.)

2. *Second way: The self-force.*† We shall now find directly the force which the field produced by the charge exerts on the charge itself. If the density of the charge is given by a function ρ and the field produced by this charge is given by $\mathbf{E}_s, \mathbf{H}_s$, the self-force is given by § 1 eq. (4)

$$\mathbf{K}^s = \int \rho \, d\tau \left(\mathbf{E}_s + \frac{1}{c}[\mathbf{v}\mathbf{H}_s]\right). \tag{8}$$

(We write the index s on the top because (8) will not be quite identical with (6).) For the field we may not of course insert the field of a point-charge since it is the field inside the particle with which we are dealing. We must first assume a certain charge distribution; in the final result the transition to a point-charge can be made.

(8) can be evaluated in the following way. We consider two charge elements *de* and *de'* at a distance $\mathbf{r}$ and determine the force which the field produced by *de* exerts on the element *de'*. For this field we may take the field of a point-charge *de'* as calculated in § 3. The total self-force is then obtained by integrating over all charge elements *dede'* (and multiplying by $\frac{1}{2}$ since each pair occurs twice). The magnetic field will not give any contribution to the self-force, and we shall omit it in the following considerations.

For the calculation we make the following simplifying assumptions:

(1) The charge distribution is *rigid* (for small velocities; for high velocities we have to assume a Lorentz contraction). For a given instant of time all charge elements will then have the same velocity and acceleration.

(2) The charge distribution is to be spherical, with a finite extension of the order of magnitude of the characteristic radius $\bar{r}_0$ (electronic radius). We shall discuss in subsection 3 in which sense $\bar{r}_0$ can afterwards be put equal to zero.

Care must be taken in applying the formulae § 3 (10) for the field produced by *de*. These formulae give the field of a point-charge at a *fixed* point P which is held at rest. But in our case, the charge element *de'*, the field within which is under consideration, is rigidly connected with the charge element *de* which produces the field, both having the same velocity at any instant. The formulae § 3 (10) can, therefore, only be applied to our problem if we put the velocity $\mathbf{v}$ of the charge equal to zero at the time t at which the self-force is to be calculated, i.e. for a particle at rest: thus,

$$\mathbf{v}(t) = 0. \tag{9}$$

† Cf. H. A. Lorentz, *Theory of the Electron*, Leipzig, 1916.

This does not, however, mean that the velocity vanishes at the retarded time $t' = t - r/c$. (9) does not represent any real restriction of our results, since the self-force of a moving particle can always be obtained by a Lorentz transformation.

The electric field produced by de at the position of de' at the time t is given by § 3 eq. (10 a)

$$d\mathbf{E}_s(t) = \frac{de'}{s^3}\left\{\frac{1}{c^2}\left[\mathbf{r}\left[\mathbf{r}+\frac{\mathbf{v}(t')}{c}r, \dot{\mathbf{v}}(t')\right]\right] - \left(1-\frac{v^2(t')}{c^2}\right)\left(\mathbf{r}+\frac{\mathbf{v}(t')}{c}r\right)\right\} \quad (10)$$

$$s = r+\left(\frac{\mathbf{v}(t')}{c}\mathbf{r}\right), \qquad \mathbf{v}(t) = 0, \qquad t' = t-\frac{r}{c}.$$

Here the vector $\mathbf{r}$ is constant. $\dot{\mathbf{v}}$ can therefore also be considered as the derivative with respect to t instead of $t-r/c$. The expression (10) has to be integrated over the charge distribution. In this integral one has again to insert a different retarded time $t-r/c$ for each de'. In general the self-force would depend essentially upon the motion at all previous times. We obtain a reasonable result only if we make a further assumption about the *motion* of the particle which is very fundamental for the whole radiation theory:

(3) *The motion of the particle varies slowly*, so that the change of its acceleration within the time which light needs to pass the charge distribution is smaller than the acceleration itself.

This interval of time being $\bar{r}_0/c$, our assumption becomes

$$\frac{\bar{r}_0}{c}\ddot{\mathbf{v}} \ll \dot{\mathbf{v}}. \quad (11)$$

(11) represents not only a condition which the external forces must satisfy, but also a restriction on the self-force itself. If the self-force were so strong that (11) did not hold our theory would break down entirely.

Assuming the condition (11), the effect of retardation in equation (10) becomes a small correction. We can then develop (making use of (9))

$$\left.\begin{aligned}\dot{\mathbf{v}}(t') &= \dot{\mathbf{v}}(t)-\frac{r}{c}\ddot{\mathbf{v}}(t)+\ldots \\ \mathbf{v}(t') &= -\dot{\mathbf{v}}(t)\frac{r}{c}+\frac{1}{2}\frac{r^2}{c^2}\ddot{\mathbf{v}}(t)+\ldots,\end{aligned}\right\} \quad (12)$$

where in both equations the second term is small compared with the first one.

(12) has to be inserted in (10). Neglecting all terms containing $(r/c)^4$ and higher powers, we can neglect in the first term of (10) $\mathbf{v}r/c$ and replace s by r. In the second term v^2/c^2 can be neglected, and for $1/s^3$ we may write

$$\frac{1}{s^3} = \frac{1}{r^3}\left(1+3\frac{(\mathbf{r}\dot{\mathbf{v}}(t))}{c^2}-\frac{3}{2}\frac{(\mathbf{r}\ddot{\mathbf{v}})r}{c^3}\right). \tag{13}$$

Inserting (12) and (13) in (10) we obtain

$$d\mathbf{E}_s(t) = de'\left[-2\frac{\mathbf{r}(\mathbf{r}\dot{\mathbf{v}})}{r^3c^2}+\frac{\mathbf{r}(\ddot{\mathbf{v}}\mathbf{r})}{2r^2c^3}-\frac{\mathbf{r}}{r^3}+\frac{\ddot{\mathbf{v}}}{2c^3}\right]. \tag{14}$$

Since we have to integrate (14) over a spherical charge distribution, we may at once take the average over all directions of $\mathbf{r}$. The third term then vanishes, representing simply the electrostatic force which gives of course no contribution to the self-force. In the other terms only the components of $\mathbf{r}$ in the directions of $\dot{\mathbf{v}}$ or $\ddot{\mathbf{v}}$ respectively give a contribution. The average becomes

$$\overline{d\mathbf{E}_s(t)} = de'\left(-\frac{2}{3}\frac{\dot{\mathbf{v}}}{rc^2}+\frac{2}{3}\frac{\ddot{\mathbf{v}}}{c^3}\right). \tag{15}$$

Integrating over all charge elements de, de' we obtain the self-force

$$\mathbf{K}^s = \mathbf{K}_0+\mathbf{K}_s \tag{16a}$$

$$\mathbf{K}_0 = -\frac{2}{3}\frac{\dot{\mathbf{v}}}{c^2}\frac{1}{2}\int\frac{dede'}{r} = -\frac{2}{3}\frac{\dot{\mathbf{v}}}{c^2}\mu_0 \tag{16b}$$

$$\mathbf{K}_s = \frac{2}{3}\frac{\ddot{\mathbf{v}}}{c^3}\frac{1}{2}\int dede' = \frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{v}}. \tag{16c}$$

We discuss first the second term $\mathbf{K}_s$. This is *independent of the charge distribution* and identical with the self-force deduced in subsection 1 eq. (6) from the energy-balance. Thus, the damping force (6) which simply takes account of the energy balance, represents the second approximation—due to the *retardation inside the particle*—to the self-force.

Comparing (16c) with the condition (11), which had to be imposed on the motion of the particle, this condition becomes:

$$\mathbf{K}_s \ll \frac{2}{3}\frac{e^2}{\bar{r}_0c^2}\dot{\mathbf{v}} = \frac{2}{3}m\dot{\mathbf{v}}\frac{r_0}{\bar{r}_0}. \tag{17}$$

We shall later put $\bar{r}_0$ equal to zero. The conditions (11), (17) are then satisfied automatically. The condition (7) used in the derivation of the previous subsection is then only required if the reaction force

is to be considered as a small effect (which is the case in all applications) but not as a matter of principle.

If we take the expansion (12) to the next power of r/c, we obtain further terms of $\mathbf{K}_s$, the next one being of the order

$$\mathbf{K}' \sim \frac{\dddot{\mathbf{v}}}{c^4} \int r \, dede' \sim \frac{e^2 \dddot{\mathbf{v}}}{c^4} \bar{r}_0. \tag{18}$$

This term depends upon the structure of the electron (the same is true for all higher terms). It is, however, proportional to $\bar{r}_0$ and vanishes for a point electron. For a harmonic oscillator the ratio of the two subsequent terms K_s/K' is of the order $\bar{r}_0/\lambda$. (12) or (16) can therefore be considered as an expansion in powers of the ratio $\bar{r}_0/\lambda$. Only for $\bar{r}_0/\lambda \ll 1$ (equation (7')) is such an expansion possible.

Thus, the condition that the self-force is small is a very fundamental one for the radiation theory. Without this condition the self-force (with the higher approximations in the development (16)) would depend essentially upon the *structure* of the particle; we should not expect such a dependence to have any physical meaning. For $\bar{r} = 0$ the self-force will always be small.

3. *Self-energy.* The first term of the self-force (16 b) is proportional to the *acceleration*. It therefore takes the same form as the *inertia force* $m\dot{\mathbf{v}}$. The factor $\frac{1}{2} \int dede'/r = \mu_0$ represents the electrostatic *self-energy* or the energy contained in the static field of the particle. The latter depends upon the structure of the particle, and would become *infinite* for a point-charge. For a charge with the extension $\bar{r}_0$ it is of the order of magnitude

$$\mu_0 = \frac{e^2}{\bar{r}_0}. \tag{19}$$

This term cannot by any method be distinguished from the inertia term. Since we know nothing about the nature of the inertial mass m, we could just as well take the two terms together and assume that the self-force $\mathbf{K}_0$ is contained in the definition of the mass m. In Abraham's theory it was even assumed that no 'non-electromagnetic' mass exists (see § 2. 4). This procedure is, however, faced by a great difficulty. For a moving particle we have as well as a self-energy a self-momentum also, and we should assume that this self-momentum is also contained in the definition of the momentum of the particle. But this is hardly possible since we have seen in § 2 that the energy and momentum of the field of the particle have not the same trans-

formation character as the energy and momentum of the particle itself. They do not form a 4-vector. One could overcome these difficulties only if one were to introduce other forces of a non-electromagnetic character in the dynamics of a particle which could compensate in some way the wrong transformation properties of the field.

The necessity for the introduction of special non-electromagnetic forces in the theory of an electron is also evident from the fact that a charge distribution can never be stable under pure electromagnetic forces, since the different parts of the charge repel each other.

If $\mathbf{K}_0$ (16 a) is to be finite, the particle must have a finite extension. If, for instance, $\mathbf{K}_0$ is put equal to the observed inertia $m\dot{\mathbf{v}}$, $\bar{r}_0$ would be of the order of r_0:

$$\bar{r}_0 \sim r_0 = \frac{e^2}{mc^2},$$

$$\mu_0 \sim \mu \equiv mc^2. \tag{20}$$

For an electron r_0 becomes (from the observed mass and charge)

$$r_0 = 2{\cdot}80 \times 10^{-13} \text{ cm}. \tag{21}$$

However, in the Maxwell-Lorentz theory it has not been possible, for the reasons mentioned above, to introduce a finite size of the elementary particles. The quantum theory also makes essentially use of the idea of the point-charge. To overcome, then, the difficulty of the *infinite self-energy* two views are possible: (i) one might try to modify the Maxwell equations at distances of the order r_0 from the point-charge so that $\mathbf{K}_0$ becomes finite. An attempt in this direction is the non-linear field theory of Born and Infeld.† (ii) An alternative way is to *omit the self-energy* entirely, but, of course, assume the observed inertia $m\dot{\mathbf{v}}$. $\bar{r}_0$ can then be put equal to zero throughout and the only non-vanishing part of the self-force is the term (16 c) proportional to $\ddot{\mathbf{v}}$, which is independent of $\bar{r}_0$. If this term of the self-force is taken into account, no further restriction of the validity of the theory such as (11) exists. We shall adopt this procedure throughout and treat the electron as a *point-charge*. In § 25 it will be shown that the omission of the self-energy can be carried out in a Lorentz-invariant way.

A more deeply founded theory on these lines is due to Dirac,† but in this book we confine ourselves to the simple omission method.

† References on p. 253

4. *Line breadth.* Let us consider now the effect which the reaction force (16 c) has on the motion of the particle. The change in the motion due to this reaction force will of course also affect the radiation emitted by the particle.

We consider again a linear harmonic oscillator as a simple model for a light source. If we neglect the reaction force, the oscillator will vibrate for an infinitely long time. But because of the damping force the amplitude of the oscillator will decrease. The equation of motion is according to (16 c)

$$m\ddot{x} = -m\nu_0^2 x + \frac{2}{3}\frac{e^2}{c^3}\dddot{x}. \tag{22}$$

Since we have seen that the reaction force is small we can as a first approximation for $\dddot{x}$ insert the value for the undamped motion $-\nu_0^2\dot{x}$. Thus (22) becomes

$$\ddot{x} = -\nu_0^2 x - \gamma\dot{x} \tag{23}$$

$$\gamma = \frac{2}{3}\frac{e^2\nu_0^2}{mc^3} = \frac{2}{3}\frac{\nu_0^2}{c}r_0 \ll \nu_0. \tag{24}$$

The solution of (23) is approximately ($\gamma \ll \nu_0$)†

$$x = x_0\, e^{-\gamma t/2} e^{i\nu_0 t}. \tag{25}$$

The energy of the oscillator averaged over one period is

$$W = \tfrac{1}{2}m(\dot{x}^2 + \nu_0^2 x^2) = W_0\, e^{-\gamma t}. \tag{26}$$

Thus the energy decreases exponentially, $1/\gamma$ representing the time taken to decrease in the ratio $e : 1$. We therefore call $1/\gamma$ the *lifetime* of the oscillator. The condition (24) ($\gamma \ll \nu_0$) expresses the fact that this time is long compared with one period; otherwise the motion would not be even approximately periodic. γ is only a function of the frequency and does not depend upon the amplitude of the oscillator.

The light which is emitted by such an oscillator has an amplitude which is proportional to $\ddot{x}$, i.e. to $-\nu_0^2 x$. It decreases with t, therefore, in the same way as the amplitude of the oscillator, so that

$$\mathbf{E} = \mathbf{E}_0\, e^{-\gamma t/2} e^{i\nu_0 t}. \tag{27}$$

(27) no longer represents a monochromatic wave but a wave with

† If we write for a classical wave an exponential function, the real part has always to be understood.

a certain intensity distribution $I(\nu)$. To obtain this intensity distribution we develop (27) in a Fourier series:

$$\mathbf{E} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{E}(\nu) e^{i\nu t}\, d\nu, \qquad \mathbf{E}(\nu) = \frac{1}{2\pi}\mathbf{E}_0 \int_0^{\infty} e^{i(\nu_0-\nu)t} e^{-\gamma t/2}\, dt$$

or

$$\mathbf{E}(\nu) = \frac{1}{2\pi}\mathbf{E}_0 \frac{1}{i(\nu_0-\nu)-\gamma/2}.$$

The intensity distribution becomes

$$I(\nu) \simeq |E(\nu)|^2 = I_0 \frac{\gamma}{2\pi} \frac{1}{(\nu-\nu_0)^2+\gamma^2/4}. \tag{28}$$

The factor I_0 has been chosen so that the total intensity $\int I(\nu)\, d\nu$

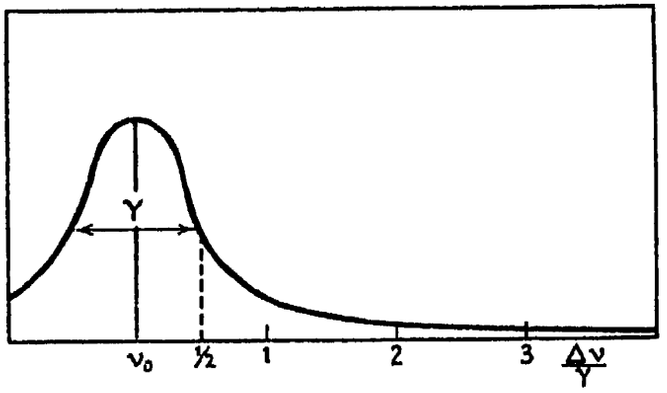

Fig. 2. Natural line breadth.

is equal to I_0. The line represented by equation (28) has a maximum intensity for the frequency ν_0, i.e. for the frequency of the undamped oscillator. It falls off on both sides rather slowly, roughly as $1/(\nu_0-\nu)^2$. For $\nu_0-\nu = \gamma/2$ the intensity is half of the maximum intensity at $\nu = \nu_0$. We therefore call γ also the *breadth at half maximum* (see Fig. 2).

The breadth of the line is equal to the reciprocal *lifetime*. If we express the line breadth in wave-length units we have according to (24)†

$$\Delta(2\pi\lambda) = 2\pi c \frac{\Delta\nu}{\nu_0^2} = 2\pi \frac{c\gamma}{\nu_0^2} = \frac{4\pi}{3} r_0 = 1{\cdot}17 \times 10^{-4}\,\text{Å.U.} \tag{29}$$

The line breadth is then independent of the frequency and equal to the electronic radius r_0 (besides a numerical factor) which is a universal constant.

In the quantum theory we shall treat the problem of the line breadth in § 12 where we shall also discuss the question from the experimental point of view.

† We denote the wave-length by $2\pi\lambda$.

5. *Broadening by collisions.* Besides the natural damping due to the emission of light itself there are many other reasons for *broadening* of a spectral line (see § **12.** 4). We shall here, while discussing the classical theory, only consider shortly the effect of broadening by *collisions*. We assume that our oscillator is subjected to collisions which disturb its vibration at time-intervals τ, say. These time-intervals τ may be distributed according to a statistical law, the number having values between τ and $\tau+d\tau$ being

$$w(\tau) = \tfrac{1}{2}\Gamma e^{-\Gamma\tau/2} \tag{30}$$

with an average value $\bar{\tau} = 2/\Gamma$. We can describe the effect of such a collision, following Lorentz,† by assuming that the phase of the vibration, and therefore of the emitted wave, is shifted by an arbitrary amount at each collision. Such a wave is no longer monochromatic. The Fourier development, similar to that of equation (28) yields, neglecting the natural breadth γ:

$$\mathbf{E}(\nu) = \frac{\mathbf{E}_0}{2\pi}\int_0^\tau e^{i(\nu_0-\nu)t}\,dt = \frac{E_0}{2\pi}\frac{e^{i(\nu_0-\nu)\tau}-1}{i(\nu_0-\nu)}. \tag{31}$$

The intensity $|E(\nu)|^2$ averaged over all values of τ according to the distribution (30) is then given by

$$\overline{|E(\nu)|^2} = \int_0^\infty w(\tau)\,|E(\nu)|^2\,d\tau = \frac{E_0^2}{2\pi^2}\frac{1}{(\nu_0-\nu)^2+\Gamma^2/4}. \tag{32}$$

The line has therefore the same shape as the natural line (28). The breadth at half maximum is equal to Γ, i.e. to twice the average time between two collisions. By analogy with the radiative lifetime $1/\gamma$, we may call $1/\Gamma$ the lifetime due to collisions.

5. Scattering, absorption

The Maxwell-Lorentz theory gives an account not only of the emission of light by a moving charge but also of the motion which an external field, say a light wave, impresses on a particle (Lorentz force § **1** eq. (5)). The wave will, in general, transfer energy to the particle and will therefore be to some extent *absorbed*. The impressed motion of the particle may also give rise to the emission of a secondary light wave (*scattering*). Since a knowledge of the classical results is important for a critical understanding of those of the quantum theory, we shall consider these radiation processes shortly here.

† H. A. Lorentz, *Theory of the Electron*, Leipzig, 1916.

1. *Scattering by free electrons.* A light wave with the frequency ν and the field-strength

$$\mathbf{E} = \mathbf{E}_0 \cos(\nu t + \delta)$$

may fall upon a free electron, the mean position of which is at rest. If we neglect the reaction force § 4 eq. (6) and all relativistic effects (including the magnetic force) the equation of motion becomes

$$m\ddot{\mathbf{x}} = e\mathbf{E}_0 \cos(\nu t + \delta) \tag{1}$$

with the solution

$$\mathbf{x} = -\frac{e\mathbf{E}_0}{m\nu^2} \cos(\nu t + \delta). \tag{2}$$

The electron performs therefore a vibration with the same frequency as the incident wave and with a phase difference π. This gives rise to the emission of a secondary wave of the same frequency. The time average value of the intensity, according to § 3 eq. (23), at a distance R from the electron is

$$I = |\mathbf{S}| = \frac{e^2 \overline{\ddot{x}^2} \sin^2\Theta}{4\pi R^2 c^3} = \frac{e^4 \sin^2\Theta \, E_0^2}{8\pi R^2 m^2 c^3}, \tag{3}$$

where Θ is the angle between the direction of observation ($\mathbf{R}$) and the direction of polarization of the incident wave $\mathbf{E}_0$. Introducing the intensity of the primary radiation $I_0 = cE_0^2/8\pi$ we have

$$I = \phi(\Theta) \frac{I_0}{R^2}, \qquad \phi(\Theta) = r_0^2 \sin^2\Theta, \tag{4}$$

where r_0 is the classical electronic radius and ϕ a quantity with the dimensions cm.^{2} which we call the *cross-section* for *scattering*.

If the primary wave is unpolarized we have to average over Θ and obtain

$$\overline{\sin^2\Theta} = \tfrac{1}{2}(1 + \cos^2\theta)$$

$$I = \frac{I_0}{R^2} \tfrac{1}{2} r_0^2 (1 + \cos^2\theta), \tag{4'}$$

where θ represents the angle of scattering.

The total scattered radiation is obtained from (4') by integrating over all angles, the total cross-section becoming

$$\phi = \frac{8\pi}{3} r_0^2 = 6{\cdot}57 \times 10^{-25} \text{ cm.}^2 \tag{5}$$

Thus the cross-section for scattering by a free electron is a *universal constant* and independent of the primary frequency.

The formulae (4) and (5) were first deduced by J. J. Thomson. We shall see that they hold also in the quantum theory as long as relativistic effects can be neglected.

2. *Scattering by an oscillator.* As a second case we shall consider the scattering of light of frequency ν by an elastically bound electron with the frequency ν_0. It is important here to take into account also the damping force § 4 eq. (6). If the electron performs only the free vibration but no forced vibration, due to the light wave, its motion will be periodic with the frequency ν_0. Therefore, we can write for the damping force

$$\frac{2}{3}\frac{e^2}{c^3}\dddot{\mathbf{x}} = -m\gamma\dot{\mathbf{x}}, \qquad \gamma = \frac{2}{3}\frac{e^2\nu_0^2}{mc^3} = \frac{2}{3}r_0\frac{\nu_0^2}{c}. \tag{6}$$

The equation of motion becomes then

$$\ddot{\mathbf{x}}+\gamma\mathbf{x}+\nu_0^2\mathbf{x} = \frac{e}{m}\mathbf{E}_0 e^{i\nu t} \tag{7}$$

with the solution $\mathbf{x} = \mathbf{x}_0 e^{i\nu t}$, $\qquad \mathbf{x}_0 = \dfrac{\mathbf{E}_0\, e/m}{\nu_0^2-\nu^2+i\nu\gamma}$. (8)

The intensity of the radiation scattered per unit time and cm.^{2} in a distance R becomes (for $\ddot{x}$ we have to insert the real part):

$$I = \frac{e^2\sin^2\theta}{4\pi R^2c^3}\ddot{x}^2 = \frac{c}{4\pi R^2}E_0^2r_0^2\frac{\nu^4\sin^2\theta}{(\nu_0^2-\nu^2)^2+\nu^2\gamma^2}\cos^2(\nu t-\delta) \tag{9}$$

$$\tan\delta = \frac{\nu\gamma}{\nu_0^2-\nu^2}.$$

The scattered radiation shows a phase displacement δ which, however, is only appreciable in the neighbourhood of $\nu = \nu_0$. Taking the average over one period and introducing the intensity of the primary radiation we obtain

$$I = \frac{I_0}{R^2}\phi(\theta), \qquad \phi(\theta) = r_0^2\frac{\nu^4\sin^2\theta}{(\nu_0^2-\nu^2)^2+\nu^2\gamma^2}. \tag{10}$$

The cross-section ϕ for the total radiation scattered is obtained by integration over the sphere, giving

$$\phi = \frac{8\pi}{3}r_0^2\frac{\nu^4}{(\nu_0^2-\nu^2)^2+\nu^2\gamma^2}. \tag{11}$$

(11) represents the well-known dispersion formula. For $\nu_0 \to 0$ and $\gamma \ll \nu$ we obtain again equation (5) for the scattering by a free electron. For frequencies far away from the resonance frequency $\gamma^2\nu^2$ may be neglected. In the neighbourhood of $\nu = \nu_0$ (11) becomes very large and we have the case of *resonance fluorescence*. We may then put $\nu \sim \nu_0$:

$$\phi = \frac{2\pi}{3}r_0^2\frac{\nu^2}{(\nu_0-\nu)^2+\gamma^2/4}. \tag{12}$$

In the quantum theory we shall meet formulae which represent a simple generalization of (11) and (12) (see §§ **14** and **15**).

3. *Absorption.* Finally, we shall consider the energy transfer from the incident wave to the oscillator. This question is a little more complicated. In the case of resonance, which is particularly interesting, we shall obtain a definite result only if we assume that the primary radiation has, in the neighbourhood of ν_0, a continuous *intensity distribution* $I_0(\nu)\,d\nu$ being the energy per cm.^{2} sec., say. Since we shall discuss the shape (breadth) of the absorption line in the quantum theory, we shall calculate here only the total absorption and can therefore neglect the reaction γ. For a single Fourier component ν the equation of motion of the oscillator is

$$\ddot{\mathbf{x}}+\nu_0^2\mathbf{x} = \frac{e}{m}\mathbf{E}(\nu)\cos(\nu t+\delta_\nu), \tag{13}$$

where δ_ν is the phase of a single wave. We assume that these phases are distributed at random.

As for a single frequency ν the energy transfer will depend essentially upon the phase difference of the oscillator and the wave we must take into account also the free vibration of the oscillator. We choose a solution of (13) so that for $t = 0$ only the free vibration is excited. This solution is

$$\mathbf{x} = \frac{e}{m}\mathbf{E}(\nu)\frac{1}{\nu_0^2-\nu^2}[\cos(\nu t+\delta_\nu)-\cos(\nu_0 t+\delta_\nu)]+\mathbf{b}\sin(\nu_0 t+\theta), \tag{14}$$

where $\mathbf{b}$ represents the amplitude and θ the phase of the oscillator at $t = 0$. The energy transfer per unit time (and per frequency interval) to the oscillator is equal to the work performed by the light wave ν.

$$\epsilon_\nu = e(\dot{\mathbf{x}}\mathbf{E}(\nu))\cos(\nu t+\delta_\nu). \tag{15}$$

If we then integrate (15) over a time τ containing an integer number of periods $1/\nu$ the term $\cos(\nu t+\delta_\nu)$ of (14) vanishes and we obtain:

$$\int_0^\tau \epsilon_\nu\,dt = \frac{e^2E^2(\nu)}{m}\frac{\nu_0}{\nu_0^2-\nu^2}\int_0^\tau dt\,\sin(\nu_0 t+\delta_\nu)\cos(\nu t+\delta_\nu)+$$

$$+e(\mathbf{E}(\nu)\mathbf{b})\nu_0\int_0^\tau dt\,\cos(\nu_0 t+\theta)\cos(\nu t+\delta_\nu). \tag{16}$$

This integral depends upon the phases and may even assume negative values. For certain phases therefore the oscillator transfers energy to the light wave (*induced emission of light*). But since the phases

δ_ν were distributed at random, we may take the average over δ_ν. Then the last term of (16) vanishes, the first term is always positive and becomes

$$\int_0^\tau \bar{\epsilon}_\nu \, dt = \frac{e^2 E^2(\nu)}{2m} \frac{\nu_0}{\nu_0^2 - \nu^2} \frac{1 - \cos(\nu_0 - \nu)\tau}{\nu_0 - \nu}. \tag{17}$$

This energy transfer is large only in the neighbourhood of the resonance frequency $\nu = \nu_0$. We may therefore put $\nu \sim \nu_0$. (17) represents the contribution of one single wave ν. We have now to integrate it over a certain frequency interval taking into account the fact that the intensity distribution was

$$I_0(\nu)\, d\nu = \frac{c}{4\pi} E^2(\nu) \overline{\cos^2(\nu t + \delta_\nu)}\, d\nu = \frac{c}{8\pi} E^2(\nu)\, d\nu.$$

In the region where (17) is appreciable we can assume $I_0(\nu)$ to be constant ($= I_0(\nu_0)$). The integration over ν can be extended from 0 to ∞, since (17) has a very strong maximum at $\nu = \nu_0$. If $\nu_0 \tau \gg 1$, the integral occurring in (17) is of the form

$$\int_{-\infty}^{+\infty} \frac{1 - \cos x}{x^2}\, dx = \pi, \tag{18}$$

when $x = (\nu_0 - \nu)\tau$. Thus the energy absorbed per unit time becomes

$$S = \frac{1}{\tau} \int_0^\infty \int_0^\tau \bar{\epsilon}_\nu \, dt d\nu = \frac{2\pi^2 e^2}{mc} I_0(\nu_0). \tag{19}$$

The energy transferred to the oscillator is on the average *proportional to the time τ, and to the incident intensity* at the resonance frequency ν_0. Apart from that it is independent of ν_0.

The quantum theory will lead to a very similar formula (§ 11).

6. The field as a superposition of plane waves. Hamiltonian form of the field equations

The theory developed in the previous paragraphs gives—within the framework of a classical theory—a complete picture of the electromagnetic field and of the mutual interaction between the field and a charged particle. As a preliminary to the extension of this theory, which is necessary to take account of quantum phenomena, it is, however, useful to express it in another form. Since the quantum theory of a particle is based essentially on the canonical form of classical dynamics it will be convenient for our purpose to express

also the classical *radiation theory* in a *canonical form*. In fact one can easily transform the whole system of field equations § 1 (2), (5) in such a way that they appear as Hamiltonian equations with a single Hamiltonian function depending on the coordinates of the particles and on some other variables which describe the field. We shall carry out this transformation in three steps.

1. *The pure radiation field.* We consider first that part of the field which is independent of the charges (light waves). According to § 1. 3, one can derive it from the vector potential **A** only, normalizing the scalar potential so that $\phi = 0$. **A** will then satisfy the equations

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \ddot{\mathbf{A}} = 0 \tag{1a}$$

$$\operatorname{div} \mathbf{A} = 0. \tag{1b}$$

A is a function defined at all points of space and time. If therefore we wish to describe **A** in terms of canonical variables, the number of such variables must necessarily be infinite. It is possible, however, to choose an *enumerable* set. For this purpose we assume that the whole radiation field is enclosed in a certain volume, for instance, a cube of volume L^3, and that it has to satisfy some boundary conditions on the surface of this volume. In order to obtain running waves as well as standing waves, we shall postulate as boundary conditions that **A** and its derivatives have the same values on two opposite planes of the volume, i.e. that

$$\mathbf{A} \text{ is periodic on the surface.} \tag{2}$$

L is to be considered large compared with the dimensions of the material system. The physical behaviour of the system will not then depend upon L. For convenience we shall choose our unit of length to be L.

With the boundary condition (2) the general solution of (1) can be represented as a series of orthogonal 'eigenwaves' ('eigenfunctions' of the differential equation (1)):

$$\mathbf{A} = \sum_{\lambda} q_\lambda(t) \mathbf{A}_\lambda(\mathbf{r}), \tag{3}$$

where $\mathbf{A}_\lambda$ depends only upon the space coordinates and q_λ only upon the time. $\mathbf{A}_\lambda$ has to satisfy the boundary condition (2). One can choose the $\mathbf{A}_\lambda$, for instance, so that they satisfy the wave equation

$$\nabla^2 \mathbf{A}_\lambda + \frac{\nu_\lambda^2}{c^2} \mathbf{A}_\lambda = 0 \tag{4a}$$

with
$$\operatorname{div} \mathbf{A}_\lambda = 0 \tag{4b}$$
$$\mathbf{A}_\lambda \text{ periodic in } L. \tag{4c}$$

Then q_λ satisfies the differential equation

$$\ddot{q}_\lambda + \nu_\lambda^2 q_\lambda = 0, \tag{5}$$

which is simply the equation for a harmonic oscillator.

The solutions of (4) represent an infinite set of orthogonal waves which we shall normalize in the following way:

$$\int (\mathbf{A}_\lambda \mathbf{A}_\mu)\, d\tau = 4\pi c^2 \delta_{\lambda\mu}. \tag{6}$$

The $\mathbf{A}_\lambda$ are simply cos and sin functions such as

$$\sqrt{(8\pi c^2)}\mathbf{e}_\lambda \cos(\boldsymbol{\varkappa}_\lambda \mathbf{r}), \qquad \sqrt{(8\pi c^2)}\mathbf{e}_\lambda \sin(\boldsymbol{\varkappa}_\lambda \mathbf{r}), \qquad |\kappa_\lambda| = \nu_\lambda / c,$$

where $\boldsymbol{\varkappa}_\lambda$ gives the direction of propagation, and the unit vector $\mathbf{e}_\lambda$ the direction of polarization which according to (4b) is always perpendicular to $\boldsymbol{\varkappa}_\lambda$. Thus (3) represents simply the Fourier expansion of $\mathbf{A}$. $\boldsymbol{\varkappa}_\lambda$ can assume a discrete set of values only, viz.

$$\kappa_{\lambda x} = \frac{2\pi}{L} n_{\lambda x}, \qquad \kappa_{\lambda y} = \frac{2\pi}{L} n_{\lambda y}, \qquad \kappa_{\lambda z} = \frac{2\pi}{L} n_{\lambda z}, \tag{7}$$

where the $n_{\lambda x}, n_{\lambda y}, \ldots$ are integers.

For each wave with given $\boldsymbol{\varkappa}$ two independent directions of polarization can be chosen arbitrarily. We could also take two circular polarized waves, but we shall not make use of this representation in this book.

In the general formulae (3)–(6) we denote different polarizations and the sin and cos functions by different indices λ.

Since the $\mathbf{A}_\lambda$ are given functions in space the field is characterized by the amplitudes q_λ. The field equations have been replaced by equations of the type (5). These can be written very simply as canonical equations; the Hamiltonian for an oscillator is

$$H_\lambda = \tfrac{1}{2}(p_\lambda^2 + \nu_\lambda^2 q_\lambda^2), \tag{8a}$$

and the Hamiltonian equations

$$\frac{\partial H_\lambda}{\partial q_\lambda} = -\dot{p}_\lambda, \qquad \frac{\partial H_\lambda}{\partial p_\lambda} = \dot{q}_\lambda = p_\lambda \tag{8b}$$

are obviously identical with (5).

The total field is described by an infinite set of canonical variables q_λ, p_λ, and a total Hamiltonian

$$H = \sum_\lambda H_\lambda. \tag{9}$$

In this way the field is represented as a system of independent oscillators. In classical dynamics H_λ would represent the energy of the oscillator. The same is true here. We shall show that the total energy of the radiation field is equal to the sum of the energies of all oscillators

$$U = \frac{1}{8\pi}\int (E^2+H^2)\,d\tau = \sum_\lambda H_\lambda. \tag{10}$$

The field strengths are given by

$$\begin{aligned} \mathbf{E} &= -\frac{1}{c}\dot{\mathbf{A}} = -\frac{1}{c}\sum_\lambda \dot{q}_\lambda \mathbf{A}_\lambda = -\frac{1}{c}\sum_\lambda p_\lambda \mathbf{A}_\lambda \\ \mathbf{H} &= \operatorname{curl}\mathbf{A} = \sum_\lambda q_\lambda \operatorname{curl}\mathbf{A}_\lambda. \end{aligned} \tag{11}$$

(11) has to be inserted in the expression for U (10). We then obtain integrals of the form $\int (\mathbf{A}_\lambda \mathbf{A}_\mu)\,d\tau$ and $\int (\operatorname{curl}\mathbf{A}_\lambda \operatorname{curl}\mathbf{A}_\mu)\,d\tau$. For the first one we can apply the orthogonality relation (6). The second one can be transformed as follows:

$$\int (\operatorname{curl}\mathbf{A}_\lambda \operatorname{curl}\mathbf{A}_\mu)\,d\tau = \oint d\sigma\,[\mathbf{A}_\lambda \operatorname{curl}\mathbf{A}_\mu]_\nu + \int (\mathbf{A}_\lambda \operatorname{curlcurl}\mathbf{A}_\mu)\,d\tau.$$

The surface integral vanishes because of the boundary condition (4 c). For the second integral on the right-hand side we make use of a formula of vector analysis curlcurl = graddiv$-\nabla^2$. We then have according to (4 a), (4 b)

$$\int (\operatorname{curl}\mathbf{A}_\lambda \operatorname{curl}\mathbf{A}_\mu)\,d\tau = \frac{\nu_\lambda^2}{c^2}\int (\mathbf{A}_\lambda \mathbf{A}_\mu)\,d\tau. \tag{12}$$

Making use now of the orthogonality equation (6) we obtain for the field energy

$$U = \tfrac{1}{2}\sum_\lambda (\dot{q}_\lambda^2+\nu_\lambda^2 q_\lambda^2) = \sum_\lambda H_\lambda. \tag{13}$$

Thus the energy of the field is equal to the sum of the energies of all the oscillators.

For applications of the radiation theory, especially to the quantum theory, it is more convenient not to represent the field by cos and sin waves but by (complex) exponential functions. Since the potential is real one can represent it by a series

$$\mathbf{A} = \sum_\lambda (q_\lambda(t)\mathbf{A}_\lambda + q_\lambda^*(t)\mathbf{A}_\lambda^*), \tag{14}$$

where the amplitude $q_\lambda(t)$ is now complex.

The solution of (4) and (5) can be written as follows:

$$\mathbf{A}_\lambda = \mathbf{e}_\lambda \sqrt{(4\pi c^2)} e^{i(\boldsymbol{\kappa}_\lambda \mathbf{r})}, \qquad |\kappa_\lambda| = \nu_\lambda / c \tag{15a}$$

$$q_\lambda = |q_\lambda| e^{-i\nu_\lambda t}. \tag{15b}$$

$q_\lambda \mathbf{A}_\lambda$ represents a wave travelling in the direction of $+\boldsymbol{\varkappa}_\lambda$. $\boldsymbol{\varkappa}_\lambda$ can again assume the values (7), where the $n_{\lambda x}$ are *positive or negative* integers. Waves with opposite directions $\boldsymbol{\varkappa}_\lambda$ and $-\boldsymbol{\varkappa}_\lambda$ are denoted by a different λ. The $\mathbf{A}_\lambda$ are orthogonal in the following sense:

$$\int (\mathbf{A}_\lambda \mathbf{A}_\mu^*)\, d\tau = \int (\mathbf{A}_\lambda \mathbf{A}_{-\mu})\, d\tau = 4\pi c^2 \delta_{\lambda\mu}, \tag{16}$$

where $\mathbf{A}_{-\mu}$ denotes the wave with the propagation vector $-\boldsymbol{\varkappa}_\mu$.

In this representation the q_λ and q_λ^* are not canonical, but one can introduce new canonical variables (which are real)

$$Q_\lambda = q_\lambda + q_\lambda^* \tag{17}$$

$$P_\lambda = -i\nu_\lambda (q_\lambda - q_\lambda^*) = \dot{Q}_\lambda.$$

The field equation (5), which holds for both q_λ and its conjugate complex, can be deduced from the Hamiltonian

$$H_\lambda = 2\nu_\lambda^2 q_\lambda q_\lambda^* = \tfrac{1}{2}(P_\lambda^2 + \nu_\lambda^2 Q_\lambda^2). \tag{18}$$

The Hamiltonian equations

$$\frac{\partial H_\lambda}{\partial Q_\lambda} = -\dot{P}_\lambda, \qquad \frac{\partial H_\lambda}{\partial P_\lambda} = \dot{Q}_\lambda \tag{19}$$

are then equivalent to (5). In the same way as before one can show that $\sum H_\lambda$ represents the total field energy U.

In the quantum theory and its applications the use of the complex variables q_λ, q_λ^* and of the Q_λ, P_λ is more convenient than the real representation (3). For the purpose of this paragraph it is, however, simpler to use the real functions.

Finally, we shall determine the number of radiation oscillators contained in the volume L^3 with a given polarization, a given direction of propagation (within an element of the solid angle $d\Omega$), and a given frequency between ν and $\nu + d\nu$. According to (7), this number must be equal to the volume-element in n-space ($n_{\lambda x}$ being integers), at least if we assume that the wave-length c/ν is small compared with L. Since ν is given by

$$\nu_\lambda^2 = \left(\frac{2\pi c}{L}\right)^2 (n_{\lambda x}^2 + n_{\lambda y}^2 + n_{\lambda z}^2) \tag{20}$$

the volume element in n-space is given by

$$\rho_\nu d\nu d\Omega\, L^3 = n^2\, dn d\Omega = \nu^2\, d\nu d\Omega\, L^3/(2\pi c)^3.$$

This expression is proportional to the volume L^3. It is actually independent of the shape of the enclosure. The number of radiation oscillators per unit volume is therefore equal to

$$\rho_\nu d\nu d\Omega = \frac{\nu^2\, d\nu d\Omega}{(2\pi c)^3}. \tag{21}$$

We call ρ_ν the *density function* for the light waves.

2. *Hamiltonian of a particle.* As a next step we consider the relativistic equation of motion of a charged particle in a given field (§ **2** eq. (35)). To write this equation in Hamiltonian form, it is only necessary to use the fact that the Hamiltonian represents the total energy E of the particle. In § **2** eq. (37) we saw that E represents the 4th component of a 4-vector p_i. Now E must be expressed as a function of the canonical coordinates and momenta. For a Cartesian system of coordinates the canonical momenta are identical with the ordinary momenta. Then equation (38) in § **2** gives the required relation between $p_4 = iE$ and the momenta $p_1 = p_x$, etc. This equation can be written as follows:

$$H \equiv E = e\phi + \sqrt{\{\mu^2 + (\mathbf{p} - e\mathbf{A})^2\}}, \qquad \mu = mc^2. \tag{22}$$

We are thus led to expect that (22) will represent the correct Hamiltonian. In fact, we obtain (using energy units for the momenta, compare § **1**. 4)

$$\frac{\partial H}{\partial p_x} = \frac{1}{c}\dot{q}_x \equiv \frac{v_x}{c} = \frac{p_x - eA_x}{\sqrt{\{\mu^2 + (\mathbf{p} - e\mathbf{A})^2\}}} \tag{23 a}$$

$$\frac{\partial H}{\partial x} = -\frac{\dot{p}_x}{c} = e\frac{\partial \phi}{\partial x} - \frac{e}{c}\left(v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x}\right). \tag{23 b}$$

On the other hand, the total derivative of A_x, with respect to t, is given by

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z}. \tag{24}$$

Adding (24) to (23 b) we obtain with the definition of the potentials given in § **1** (7 a) and (7 b)

$$\frac{1}{c}\frac{d}{dt}(p_x - eA_x) = e\left(E_x + \frac{1}{c}[\mathbf{vH}]_x\right). \tag{25}$$

(25) is identical with the equation of motion § **2** eq. (35), since $p_i - eA_i = u_i$ represents the 4-vector of the kinetic momentum,

(23 a) gives the correct connexion between u_1 and v_x since the equations § 2 (33) and (37),

$$u_1 = \frac{mcv_x}{\sqrt{(1-\beta^2)}}, \qquad \beta = \frac{v}{c},$$

can also be written in the form (23 a).

Thus (22) represents the correct Hamiltonian for a particle. It contains the interaction of the particle with the field. Just as in the expression for the Lorentz force, the field which must be inserted in (22) is the external field produced by magnets, condensers, light sources, etc., as well as the field produced by the charge itself. The latter gives the reaction of the field to the moving charge. (See § 4.)

If the momentum is small compared with the rest-energy μ we may take for (22) the non-relativistic approximation (omitting the constant term μ)

$$H = e\phi + \frac{(\mathbf{p}-e\mathbf{A})^2}{2\mu}. \qquad \text{N.R. (26)}$$

This is the ordinary non-relativistic energy function for a particle in a field having the potentials ϕ and $\mathbf{A}$. In a pure electrostatic field ($\mathbf{A} = 0$) the second term reduces to the kinetic energy $p^2/2\mu$.

3. *General system of particles and field.*† Hitherto we have transformed to the Hamiltonian form the equations of motion of a particle and of a field consisting purely of light waves. As our last step, we must now consider a general system consisting of a particle and a field of any type. In order to obtain a relativistic theory also for the interaction of two particles, we shall assume there to be several particles of charges e_k. Each particle is then described by a set of canonical variables, q_k, p_k say, with Hamiltonian

$$H_k = e_k\phi(k) + \sqrt{[\mu_k^2 + \{\mathbf{p}_k - e\mathbf{A}(k)\}^2]}, \qquad (27)$$

where $\phi(k)$, $\mathbf{A}(k)$ represent the field at the position of the kth particle. The total Hamiltonian for all particles becomes

$$H = \sum H_k \qquad (28\,a)$$

$$\frac{1}{c}\dot{p}_k = -\frac{\partial H}{\partial q_k}, \qquad \frac{1}{c}\dot{q}_k = \frac{\partial H}{\partial p_k}. \qquad (28\,b)$$

In (28 a) no interaction between the particles has as yet been assumed; it will be seen that the latter is contained in the Hamiltonian for the field.

† For the following sections see: E. Fermi, *Rev. Mod. Physics*, 4 (1932), 131, or H. Weyl, *Gruppentheorie und Quantenmechanik*, 2nd edition, Leipzig, 1933.

The field which has to be inserted in (27) consists of the external field produced by charges which do not belong to the system considered, and the field produced by all particles including the kth particle itself. It is, however, convenient to separate the external field ϕ^e, $\mathbf{A}^e$. Since this occurs in (27) only as a given potential energy (and potential momentum) for each particle and does not depend upon the position of the particles we may, for the following discussion, take the terms $\mathbf{A}^e$, ϕ^e together with p_k, H_k, simply writing p_k, H_k instead of $p_k - e\mathbf{A}^e$, $H_k - e\phi^e$. In the final result equation (52) we shall insert again the correct field.

The general field satisfies the equations

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} = -\frac{4\pi}{c}\rho\mathbf{v} \tag{29a}$$

$$\nabla^2\phi - \frac{1}{c^2}\ddot{\phi} = -4\pi\rho \tag{29b}$$

$$\operatorname{div}\mathbf{A} + \frac{1}{c}\dot{\phi} = 0. \tag{29c}$$

We have to express these equations in canonical form. For this purpose we assume again that the field is enclosed in a volume L^3 and that *all potentials* (and their derivatives) satisfy the boundary condition:

$$\mathbf{A}, \phi \text{ periodic on the surface.} \tag{30}$$

We then develop the potentials into a series of Fourier components as before.

The special case of a transverse field for which $\operatorname{div}\mathbf{A} = \phi = 0$ has been treated in subsection 1. For the general case $\operatorname{div}\mathbf{A} \neq 0$ we notice that every vector-field can be divided into two parts, of which the first has a divergence equal to zero and the second is the gradient of a scalar field. Thus

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2, \qquad \operatorname{div}\mathbf{A}_1 = 0, \qquad \mathbf{A}_2 \sim \operatorname{grad}\psi. \tag{31}$$

$\mathbf{A}_1$ is identical with the transverse field

$$\mathbf{A}_1 = \sum_\lambda q_\lambda \mathbf{A}_\lambda.$$

$\mathbf{A}_2$ can be developed in a similar way

$$\mathbf{A}_2 = \sum_\sigma q_\sigma(t)\mathbf{A}_\sigma, \tag{32}$$

where the $\mathbf{A}_\sigma$ satisfy the wave equation and boundary condition

$$\nabla^2\mathbf{A}_\sigma + \frac{\nu_\sigma^2}{c^2}\mathbf{A}_\sigma = 0, \qquad \mathbf{A}_\sigma \text{ periodic.} \tag{33a}$$

According to (31) $\mathbf{A}_\sigma$ can be represented as the gradient of a scalar function

$$\mathbf{A}_\sigma = \frac{c}{\nu_\sigma}\operatorname{grad}\psi_\sigma, \qquad \operatorname{curl}\mathbf{A}_\sigma = 0, \tag{33b}$$

where the ψ_σ obviously satisfy the same equation (33a). The factor c/ν_σ has been added for reasons of normalization. The $\mathbf{A}_\sigma$ represent also a set of orthogonal waves which satisfy the wave equation and boundary condition. We shall now prove that they are also orthogonal to all transverse waves $\mathbf{A}_\lambda$. For this purpose we apply a general formula of vector calculus,

$$\int d\tau[(\operatorname{curl}\mathbf{a}\operatorname{curl}\mathbf{b})+\operatorname{div}\mathbf{a}\operatorname{div}\mathbf{b}+(\mathbf{a}\nabla^2\mathbf{b})] = \oint d\sigma\{[\mathbf{a}\operatorname{curl}\mathbf{b}]_\nu+\mathbf{a}_\nu\operatorname{div}\mathbf{b}\}$$

(ν = component normal to the surface). Inserting $\mathbf{b} = \mathbf{A}_\lambda$, $\mathbf{a} = \mathbf{A}_\sigma$, the surface integral vanishes, because of the boundary conditions. The first and second terms on the left-hand side vanish by (4b) and (33b) respectively. Thus

$$\int (\mathbf{A}_\sigma\nabla^2\mathbf{A}_\lambda)\,d\tau = \int (\mathbf{A}_\sigma\mathbf{A}_\lambda)\,d\tau = 0. \tag{34}$$

The $\mathbf{A}_\lambda$ and $\mathbf{A}_\sigma$ represent a *complete* set of orthogonal waves, satisfying the wave equation and boundary condition (but not $\operatorname{div}\mathbf{A} = 0$).

In contrast to the transverse waves $\mathbf{A}_\lambda$, the $\mathbf{A}_\sigma$ represent *longitudinal waves*. We shall see that they give the *static* interaction of the particles.

In the same way one can develop the scalar potential

$$\phi = \sum_\sigma a_\sigma(t)\phi_\sigma \tag{35a}$$

$$\nabla^2\phi_\sigma+\frac{\nu_\sigma^2}{c^2}\phi_\sigma = 0, \qquad \phi_\sigma \text{ periodic.} \tag{35b}$$

These ϕ_σ must be identical with the scalar functions ψ_σ introduced in (33b) since they satisfy the same equation and boundary condition. Thus we can write

$$\mathbf{A}_\sigma = \frac{c}{\nu_\sigma}\operatorname{grad}\phi_\sigma. \tag{36}$$

Both ϕ_σ and $\mathbf{A}_\sigma$ are then normalized in the same way

$$\int \phi_\sigma\phi_\rho\,d\tau = \int (\mathbf{A}_\sigma\mathbf{A}_\rho)\,d\tau = 4\pi c^2\,\delta_{\rho\sigma}. \tag{37}$$

The coefficients $a_\sigma(t)$ and $q_\sigma(t)$ in (35a) and (32) are not independent, since the Lorentz equation (29c) has to be satisfied. Inserting (32) and (35) into (29c) we obtain the important equation

$$\nu_\sigma q_\sigma(t) = \dot{a}_\sigma(t). \tag{38}$$

The differential equations which the amplitudes $q_\lambda, q_\sigma, a_\sigma$ have to satisfy can easily be obtained. They will not be identical with the equations of a harmonic oscillator, since the field must now satisfy the inhomogeneous wave equation (29). We insert (3), (32), (35a) into (29), multiply by $\mathbf{A}_\lambda, \mathbf{A}_\sigma, \phi_\sigma$ respectively, and integrate over space. Assuming that all charges are point charges we obtain

$$\ddot{q}_\lambda + \nu_\lambda^2 q_\lambda = \frac{1}{c} \sum_k e_k(\mathbf{v}_k, \mathbf{A}_\lambda(k)) \tag{39a}$$

$$\ddot{q}_\sigma + \nu_\sigma^2 q_\sigma = \frac{1}{c} \sum_k e_k(\mathbf{v}_k \mathbf{A}_\sigma(k)) \tag{39b}$$

$$\ddot{a}_\sigma + \nu_\sigma^2 a_\sigma = \sum_k e_k \phi_\sigma(k), \tag{39c}$$

where $\mathbf{A}_\lambda(k)$ represents the value of $\mathbf{A}_\lambda$ at the position of the kth particle. Equations (39) represent a *forced vibration* of an oscillator, the *force being due to the presence of charged particles.*

The relation (38) can be expressed as an initial condition for the solutions of equations (39). Differentiating (39c) with respect to the time, we obtain, according to (36),

$$\dddot{a}_\sigma + \nu_\sigma^2 \dot{a}_\sigma = \frac{d}{dt} \sum_k e_k \phi_\sigma(k) = \sum_k e_k(\mathbf{v}_k \operatorname{grad} \phi_\sigma(k))$$

$$= \frac{\nu_\sigma}{c} \sum_k e_k(\mathbf{v}_k \mathbf{A}_\sigma(k)),$$

and inserting (39b) $$\left(\frac{d^2}{dt^2} + \nu_\sigma^2\right)(q_\sigma \nu_\sigma - \dot{a}_\sigma) = 0. \tag{40}$$

Therefore (38) is always satisfied if, at the time $t = 0$,

$$\nu_\sigma q_\sigma = \dot{a}_\sigma \quad \text{and} \quad \nu_\sigma \dot{q}_\sigma = \ddot{a}_\sigma. \tag{41}$$

If we consider only solutions which satisfy the initial conditions (41) we may consider all oscillators q_σ and a_σ as *independent.*

The differential equations (39) can easily be written in canonical form. The forces on the right-hand side are due to the particles and can therefore only be obtained from a term in the Hamiltonian which depends upon the variables of the particles. This term will be simply the Hamiltonian ΣH_k of the particles (equation (27)) which depends also upon the field and therefore contributes additional terms to the equations describing the field. Thus, (27) contains the interaction between field and particle.

For each oscillator (39) we shall have a Hamiltonian $\frac{1}{2}(p^2 + \nu^2 q^2)$.

The longitudinal waves are represented by two oscillators q_σ, a_σ. But since the scalar potential ϕ occurs in (27) with the opposite sign to the vector potential $\mathbf{A}$ we must take the Hamiltonian for the a_σ with a minus sign. Thus we assume for the longitudinal waves a Hamiltonian

$$H_\sigma = \tfrac{1}{2}(p_\sigma^2+\nu_\sigma^2 q_\sigma^2)-\tfrac{1}{2}(b_\sigma^2+\nu_\sigma^2 a_\sigma^2), \tag{42}$$

where b_σ represents the canonical conjugate momentum to a_σ.

For the entire system comprised by the particles and the field we shall expect a Hamiltonian

$$H = \underset{\text{particles}}{\sum_k H_k} + \underset{\substack{\text{transverse}\\ \text{waves}}}{\sum_\lambda H_\lambda} + \underset{\substack{\text{longitudinal}\\ \text{waves}}}{\sum_\sigma H_\sigma} \tag{43}$$

with the canonical equations:

$$\frac{\partial H}{\partial q_k} = -\frac{1}{c}\dot{p}_k \qquad \frac{\partial H}{\partial p_k} = \frac{1}{c}\dot{q}_k \quad \text{(particles)} \tag{44 a}$$

$$\frac{\partial H}{\partial q_\lambda} = -\dot{p}_\lambda \qquad \frac{\partial H}{\partial p_\lambda} = \dot{q}_\lambda \quad \text{(transverse waves)} \tag{44 b}$$

$$\left.\begin{aligned} \frac{\partial H}{\partial q_\sigma} &= -\dot{p}_\sigma & \frac{\partial H}{\partial p_\sigma} &= \dot{q}_\sigma \\ \frac{\partial H}{\partial a_\sigma} &= -\dot{b}_\sigma & \frac{\partial H}{\partial b_\sigma} &= \dot{a}_\sigma \end{aligned}\right\} \text{(longitudinal waves).} \qquad \begin{matrix} \text{(44 c)} \\ \text{(44 d)} \end{matrix}$$

In fact (44) is entirely equivalent to the field equations (39) and the equation of motion of the particles (25). We may, for instance, prove that (39 c) is identical with (44 d). a_σ, b_σ occur in H_σ and H_k. According to (27), (35 a), and (42), we have

$$\frac{\partial H_k}{\partial a_\sigma} = e_k \phi_\sigma(k), \qquad \frac{\partial H_\sigma}{\partial a_\sigma} = -\nu_\sigma^2 a_\sigma,$$

$$\frac{\partial H_k}{\partial b_\sigma} = 0, \qquad \frac{\partial H_\sigma}{\partial b_\sigma} = -b_\sigma.$$

From (44 d) we obtain

$$-\dot{b}_\sigma = \ddot{a}_\sigma = -\nu_\sigma^2 a_\sigma + \sum_k e_k \phi_\sigma(k),$$

which is identical with (39 c).

We have now represented the whole system of field equations in canonical form. The physical significance of the Hamiltonian is quite clear: the first term represents the energy of the particles (kinetic+potential); the second term the energy of the light waves (see subsection 1); the third term would represent the energy of the 'longitudinal waves'. They occur of course only in the presence of

particles and have a very simple significance. They represent the *Coulomb interaction* between the particles. This will be shown in the following subsection.

4. *Interaction of particles.* We consider the terms in the Hamiltonian (43) which depend upon the variables q_σ, a_σ of the longitudinal waves. q_σ, a_σ occur in H_σ, in the scalar potential $\sum_k e_k \phi(k)$, and in $\mathbf{A}$ under the square root of H_k. Since, according to (38), $\nu_\sigma^2 q_\sigma^2 = b_\sigma^2$, we can write for the first two terms

$$H_s \equiv H_\sigma + \sum_k e_k \phi(k) = \tfrac{1}{2} \sum_\sigma (p_\sigma^2 - \nu_\sigma^2 a_\sigma^2) + \sum_\sigma \sum_k e_k a_\sigma \phi_\sigma. \tag{45}$$

Inserting for $p_\sigma = \dot{q}_\sigma = \ddot{a}_\sigma/\nu_\sigma$ the value of (39 c), we obtain

$$H_s = \frac{1}{2} \sum_\sigma \frac{1}{\nu_\sigma^2} \sum_{i,k} e_i e_k \phi_\sigma(i) \phi_\sigma(k). \tag{46}$$

This sum can easily be evaluated. We consider one of the terms of (46),

$$H_{ik} = \sum_\sigma \frac{1}{\nu_\sigma^2} \phi_\sigma(i) \phi_\sigma(k),$$

as a function of the position of the particle i and let ∇_i^2 operate upon it. We then obtain according to (35 b)

$$\nabla_i^2 H_{ik} = \sum_\sigma \frac{\nabla^2 \phi_\sigma(i) \phi_\sigma(k)}{\nu_\sigma^2} = -\frac{1}{c^2} \sum_\sigma \phi_\sigma(i) \phi_\sigma(k) = -4\pi\delta(i-k), \tag{47}$$

where $\delta(i-k) = 0$ except if the two particles i and k have the same position. The last equation (47) expresses a general property of all systems of orthogonal eigenfunctions. Thus, H_{ik} satisfies, as a function of the position of i, the Poisson equation with a singularity at the position of k. The solution must be $H_{ik} = 1/r_{ik}$, where r_{ik} represents the distance of i and k. Therefore, the part of the Hamiltonian which depends upon the longitudinal waves becomes

$$H_s = \frac{1}{2} \sum_{i,k} \frac{e_i e_k}{r_{ik}}. \tag{48}$$

(48) represents simply the *static (Coulomb) interaction between the particles.*†

The part $H_s = H_\sigma + \sum e_k \phi(k)$ of the Hamiltonian (43) can therefore be expressed as a function of the coordinates of the particles only. We can now replace the whole system of equations (44) by a new

† This deduction is due to E. Fermi, loc. cit. See also Appendix II.

system which is deduced from a new Hamiltonian depending upon the variables $q_\lambda, p_\lambda, q_k, p_k$ only. The variables q_σ of the longitudinal waves seem still to occur in the term H_k in the Hamiltonian, since there one has to insert the total vector potential **A**. But we can show that for all solutions which satisfy the initial conditions (41) the new Hamiltonian $H(q_\lambda, q_k, p_\lambda, p_k)$ is simply given by omitting the longitudinal part of the vector potential $\sum q_\sigma \mathbf{A}_\sigma$ in H_k (and, of course, inserting (48) for H_s). The variables q_σ, a_σ are then eliminated entirely and the new Hamiltonian contains the coordinates of the particles and of the transverse waves only.

To show this we write down the Hamilton-Jacobi partial differential equation for the action function $S(q_k, q_\lambda, q_\sigma, a_\sigma)$ which corresponds to the original Hamiltonian (43):

$$\frac{\partial S}{\partial t} = H(q_k p_k,\ q_\lambda p_\lambda,\ q_\sigma p_\sigma,\ a_\sigma b_\sigma) \tag{49 a}$$

$$p_k = c\frac{\partial S}{\partial q_k}, \qquad p_\lambda = \frac{\partial S}{\partial q_\lambda}, \qquad p_\sigma = \frac{\partial S}{\partial q_\sigma}, \qquad b_\sigma = \frac{\partial S}{\partial a_\sigma}. \tag{49 b}$$

The partial differential equation (49 a) can be solved immediately if we take into account the initial conditions (41) which have to be satisfied:

$$\left.\begin{aligned} S &= S_1(q_k, q_\lambda) + S_2(q_\sigma, a_\sigma, q_k) \\ S_2 &= \sum_\sigma q_\sigma \Big(\sum_k e_k \phi_\sigma(k)/\nu_\sigma - a_\sigma \nu_\sigma \Big). \end{aligned}\right\} \tag{50}$$

Here S_1 is a function of the q_k and q_λ only. We insert this solution (50) in the differential equation (49).

From (50) and (36) it follows that

$$c\frac{\partial S_2}{\partial q_k} = e_k \sum_\sigma q_\sigma \frac{\partial \phi_\sigma(k)}{\partial q_k}\frac{c}{\nu_\sigma} = e_k \sum_\sigma q_\sigma A_\sigma(k).$$

Hence we obtain for the kinetic momentum occurring in H_k:

$$\begin{aligned} p_k - eA(k) &= c\frac{\partial S_1}{\partial q_k} + c\frac{\partial S_2}{\partial q_k} - e_k \sum_\lambda q_\lambda A_\lambda(k) - e_k \sum_\sigma q_\sigma A_\sigma(k) \\ &= c\frac{\partial S_1}{\partial q_k} - e_k \sum_\lambda q_\lambda A_\lambda(k). \end{aligned} \tag{51}$$

On the right-hand side of (51) the longitudinal waves have disappeared. S is replaced by S_1 ($p_k = c\,\partial S_1/\partial q_k$) and the vector potential by its transverse part. Since S_2 does not depend explicitly upon

the time, $\partial S_2/\partial t = 0$, we can write the differential equation (49 a) as an equation for S_1 only, inserting again an external field ϕ^e, $\mathbf{A}^e$:

$$\frac{\partial S_1}{\partial t} = H\left(q_k,\, c\frac{\partial S_1}{\partial q_k},\, q_\lambda,\, \frac{\partial S_1}{\partial q_\lambda}\right) \tag{52 a}$$

$$H = \sum_k H_k + \sum_\lambda H_\lambda + \frac{1}{2}\sum_{i,k}\frac{e_i e_k}{r_{ik}} + \sum_k e_k \phi^e(k) \tag{52 b}$$

$$H_k = \left[\mu_k^2 + \left(\mathbf{p}_k - e_k \sum_\lambda q_\lambda \mathbf{A}_\lambda - e_k \mathbf{A}^e\right)^2\right]^{\frac{1}{2}}. \tag{52 c}$$

In this Hamiltonian the longitudinal waves are eliminated entirely. H_k contains only the interaction of the particle with the light waves (and an external field $\mathbf{A}^e$). From (52) one can deduce a corresponding set of canonical equations for the variables $p_k, q_k, p_\lambda, q_\lambda$ only. They are equivalent to the original set of equations (41), (43), and (44).

The physical significance of the various terms of (52 b) is clear:

1st term: kinetic energy of particles + interaction with light waves.

2nd term: energy of light waves.

3rd term: static interaction between the particles.

4th term: potential energy in external field.

It seems as if in this theory the interaction of two particles would only be the *instantaneous* Coulomb interaction, whereas we should expect to obtain a *retarded* interaction. But this is actually the case. The retarded interaction can be considered as a superposition of a momentary static one and of light waves. The effect of retardation is contained in those terms of (52 b) which depend upon the transverse waves (first and second terms); it appears in the present theory as a mutual emission and absorption of light waves between the particles. (Compare § **10**, subsection 5, and Appendix II.)

The separation of the field into a transverse part (light waves) and a Coulomb part has a very important physical significance. We have shown in § **2** that the transverse part, in its behaviour under a Lorentz transformation, has a certain similarity to a particle, since its energy and momentum form a 4-vector. For the longitudinal part this is not the case. The *transverse part* will be seen in Chapter II to consist of *light quanta* which do in fact behave also in other ways like particles. The Coulomb part remains unquantized.

In the Coulomb term (48) the summation has to be taken over all i and k. We obtain therefore also terms of the form e_k^2/r_{kk} which

become infinite. They represent the infinite self-energy (energy of the static field of a point charge). We met this difficulty in § 4 in the discussion of the self-force of a particle. According to these considerations we have to *omit* these terms—or to assume that they are already contained in the rest energy μ_k. Therefore, we have to understand the Coulomb term in (48) as

$$H_s = \frac{1}{2} \sum_{i \neq k} \frac{e_i e_k}{r_{ik}}. \tag{53}$$

Finally we show that the field strength **E** can also be expressed, in the same way as the total energy of the system, in terms of the amplitudes of the transverse waves and the *coordinates* of the *particles* only. (**H** in any case does not depend upon the longitudinal waves.) According to (36), (38), and (39 c) we obtain for the longitudinal part of **E**:

$$\mathbf{E}_{\text{long}} = -\sum_{\sigma} \left(a_\sigma \operatorname{grad} \phi_\sigma + \frac{1}{c} \dot{q}_\sigma \mathbf{A}_\sigma \right) = -\frac{1}{c} \sum_{\sigma,k} e_k \phi_\sigma(k) \mathbf{A}_\sigma \frac{1}{\nu_\sigma}. \tag{54}$$

In (54) the amplitudes q_σ, a_σ do not occur.

5. *Example: emission of light.* Classical electrodynamics in the form developed in this section will be the foundation on which we shall raise the quantum theory of radiation. Quantum electrodynamics is based upon the classical theory in the same way as the quantum dynamics is based upon the classical dynamics, i.e. in the sense of the *correspondence principle*. In order to demonstrate the close connexion between the quantum theoretical and the classical treatment of radiation phenomena we may consider as an example the emission of light by a linear harmonic oscillator, using the method developed in this paragraph. The quantum theoretical treatment of this problem will then be almost exactly the same.

The light is considered here as a system of oscillators performing forced vibrations. The amplitudes $q_\lambda(t)$ satisfy the equations (39 a), where the external force on the right-hand side is due to the electron. Neglecting the reaction of the emitted radiation on the electron, we may put for the velocity of the electron

$$\mathbf{v} = \mathbf{v}_0 \cos \nu_0 t. \tag{55}$$

The emission of light appears here as the excitation of a harmonic oscillator (light) by a periodic external force (electron)—a problem which is very similar to the absorption of light by an oscillating electron as treated in § 5.

The differential equation (39 a) becomes

$$\ddot{q}_\lambda + \nu_\lambda^2 q_\lambda = b_\lambda \cos \nu_0 t, \qquad b_\lambda = \frac{e}{c} v_0 |A_\lambda(k)| \cos \Theta, \tag{56}$$

where Θ represents the angle between the direction of polarization (**A**) and the direction of the oscillator. We take a solution of (56) for which, at the time $t = 0$, the light oscillator performs no vibration

$$q_\lambda = \frac{b_\lambda}{\nu_\lambda^2 - \nu_0^2} (\cos \nu_0 t - \cos \nu_\lambda t). \tag{57}$$

Thus only those light oscillators will be excited which have the same frequency as the electron. The energy of the light oscillator after the time t (t may contain an integral number of vibrations ν_0) is given by

$$\begin{aligned} H_\lambda(t) &= \int_0^t dt\, \dot{q}_\lambda b_\lambda \cos \nu_0 t \\ &= \frac{b_\lambda^2}{\nu_\lambda^2 - \nu_0^2} \frac{\nu_\lambda}{2} \left[\frac{1 - \cos(\nu_\lambda - \nu_0) t}{\nu_\lambda - \nu_0} + \frac{1 - \cos(\nu_0 + \nu_\lambda) t}{\nu_0 + \nu_\lambda} \right]. \end{aligned} \tag{58}$$

We are not interested in the excitation of a single light oscillator. We therefore take the summation of (58) over all oscillators having a frequency between ν and $\nu + d\nu$ and the same direction of propagation. The number of those oscillators ρ_ν is given by formula (21). Integrating (58) over a small frequency interval in the neighbourhood of ν_0 we can neglect the second term of (58) and obtain (cf. § 5. 3, eq. (18))

$$\sum_{\nu_\lambda \sim \nu_0} H_\lambda(t) = \frac{\pi b^2}{4} t \frac{\nu_0^2 d\Omega}{(2\pi c)^3} = St. \tag{59}$$

The energy of the radiation oscillators is therefore proportional to the *time*.

Inserting for b^2 its value (56) we may take the average of $|A_\lambda(k)|^2$ over all positions of the electron in space giving $4\pi c^2$. Furthermore, if we take the summation over the directions of polarization we obtain $\sin^2\theta$ instead of $\cos^2\Theta$, where θ represents the angle between the oscillator v_0 and the direction of emission. The energy transferred to the light oscillators per unit time, i.e. the energy emitted per unit time, is then equal to

$$S = \frac{e^2 \nu_0^2}{8\pi c^3} v_0^2 \sin^2\theta\, d\Omega. \tag{60}$$

This formula is identical with formula (23) of § 3 (taking into account the fact that the time average of $\overline{v^2} = v_0^2/2$).

In the same way all other radiation processes (§ 5) can be treated. One sees that this method is not more complicated than the usual method of § 3.

II

QUANTUM THEORY OF THE FIELD *IN VACUO*

7. Quantization of the radiation field

1. *Introduction.* THE classical theory as developed in Chapter I is only correct in so far as one can neglect all effects which arise from the finite value of Planck's action constant h. Before we introduce this new constant into the theory of the electromagnetic field we desire to emphasize some of the experimental and theoretical facts from which the necessity for a quantization of the field becomes evident. Historically, it was in the theory of radiation itself that a departure from the classical theory first became necessary. In the problem of radiation in thermal equilibrium with a black body the classical theory leads to the well-known 'ultra-violet difficulty', as the density of energy in the form of short waves diverges. To avoid this difficulty Planck assumed that the energy of a monochromatic wave with the frequency ν could only assume values which were an integral multiple of a certain unit proportional to the frequency

$$E = n\hbar\nu. \tag{1}$$

n being an integer (the number of light quanta), $2\pi\hbar = h$ is the universal Planck's constant. This assumption (1) leads to a correct formula for the radiation from a black body. It means of course that the classical theory, which is incompatible with such an assumption, has to a large extent to be abandoned.

The quantization (1) of a monochromatic wave, together with the law of conservation of energy, led to Bohr's well-known *frequency condition.* It is hardly necessary to emphasize that the latter is in agreement with innumerable experiments in the field of atomic physics.

Here we only mention—as an example—one of the fundamental experiments by which the quantum nature of a light wave is shown directly, namely the experiment of Compton and Simon.

A monochromatic beam of X-rays passing through a Wilson chamber is scattered by a free electron (for X-rays, the electrons contained in the gas molecules can be considered as free). The electron is then ejected in a certain direction, forming a track in the chamber. The X-ray is scattered in another direction which can

also be observed (from the electrons ejected by the scattered wave) and which is determined by the direction of the ejected electron. Now it has been found that the directions of the ejected electron and the scattered X-ray bear exactly the same relation as in an *elastic collision of two particles* according to classical mechanics (conservation of energy and momentum). The experimental result can be explained by assuming that the beam of X-rays consists of *particles* each having an energy $\hbar\nu$ (equation (1)) and momentum $\mathbf{k}$, where $k = \hbar\nu$.

On the other hand, the same beam of X-rays shows the well-known interference phenomena such as Laue scattering, interference fringes when scattered by a grating through a small angle (Thibaud), etc., which are a proof of the wave nature of X-rays.

It is this *dual nature of light*, according to which it appears *as waves and as particles*, which necessitates the quantization of light waves for its description.

The necessity for a quantization of the electromagnetic field is shown also from another more theoretical consideration: if we assume for the present the ordinary theory of *quantum mechanics* we find the quantization of the field to be logically connected with that theory. It can be shown, in fact, that the latter represents a *compatible* theory, only if the *radiation field is quantized in a way similar to the coordinates describing the motion of a particle.*

The quantum properties of a particle are contained in the uncertainty relation for the position and momentum

$$\Delta q \Delta p \sim \hbar c \tag{2}$$

(energy units for the momentum).

This relation could be disproved immediately if the classical theory were valid for a beam of light. For if this were the case one could measure the position of the particle *exactly* by a convergent beam of light (Heisenberg's γ-ray microscope) without transferring an appreciable amount of momentum to the particle, since the momentum of the light beam could be made as small as one likes. Thus, if p was measured before, one could obtain knowledge of position and momentum exceeding the limits given by equation (2). In order that (2) should hold notwithstanding the possibility of this experiment, it is necessary that there should be for the *light beam*, a similar *uncertainty relation to* (2). This will be seen (subsection 5) to be the case if the light wave is quantized. The light beam will then have a minimum momentum which is uncertain by an amount Δp if it has

a shape and frequency suitable for measuring the position of a particle with an accuracy Δq. This momentum will be transferred to the particle in a way outside the control of the experimenter, and the uncertainty relation (2) still holds after the measurement of the position of the particle.†

Thus for a consistent quantum theory it is necessary for a *light beam to satisfy the same uncertainty relation as a particle.* The field has therefore to be quantized in a way similar to the motion of a particle.

It will be seen that the uncertainty relation (2) for the radiation follows essentially from the quantization (1) of a light wave.

The quantum theory of the electromagnetic field can be developed entirely from postulate (1) or from the uncertainty relation (2). In the development of the formalism of quantum electrodynamics, it is, however, easiest to let ourselves be guided by the *formal* analogy between classical mechanics and classical electrodynamics pointed out in § 6. In that section we represented the field by a set of canonical variables and the field equations were put in the form of Hamiltonian equations deduced from the Hamiltonian § 6, eq. (52 b). The quantum of action h can then be introduced in the same way as in the ordinary quantum mechanics.

The quantum electrodynamics which we obtain in this way satisfies the *correspondence principle.* Most of the consequences to which it leads can—at least qualitatively—be understood by arguments based on this principle, a fact which we shall be able to verify throughout, in the applications to atomic physics (Chapter III). (For a criticism of the theory see § 25.)

2. *Quantization of the pure radiation field.*‡ We consider first the part of the field which can be formed by superposition of transverse

† For a detailed discussion of this experiment see N. Bohr, *Atomtheorie und Naturbeschreibung*, Berlin, 1931 or W. Heisenberg, *Die physikalischen Prinzipien der Quantentheorie*, Leipzig, 1930.

‡ The quantum theory of the pure radiation field was developed by P. A. M. Dirac, *Proc. Roy. Soc.* **114** (1927), 243, 710, and by P. Jordan and W. Pauli, *Zs. f. Phys.* **47** (1928), 151; and the quantization of the general field by the method used here by E. Fermi, *Rev. Mod. Phys.* **4** (1932), 131. The theory in this form is equivalent to the general quantum electrodynamics of W. Heisenberg and W. Pauli, *Zs. f. Phys.* **56** (1929), 1; **59** (1930), 169 (general formalism). Compare also: O. Klein, *Zs. f. Phys.* **41** (1927), 407; W. Heisenberg, *Ann. d. Phys.* **9** (1931), 338 (formalism and correspondence principle); V. Fock, *Phys. Zs. Sowj. Union*, **6** (1934), 425; L. Rosenfeld, *Ann. d. Phys.* **5** (1930), 113; L. Landau and R. Peierls, *Zs. f. Phys.* **62** (1930), 188 (light quanta and configuration space); Report on the theory: L. Rosenfeld, *Ann. Inst. Henri Poincaré* (1931), 25; *Handb. d. Physik*, 2nd edition, XXIV, 1, articles by W. Pauli and G. Wentzel; H. A. Kramers, *Hand- u. Jahrb. d. chem. Phys.* 1 (1938).

waves. This field can be derived from the vector potential **A** which, according to § 6 eq. (14), may be written as a series of plane waves (we use the complex representation)

$$\mathbf{A} = \sum_{\lambda} (q_\lambda \mathbf{A}_\lambda + q_\lambda^* \mathbf{A}_\lambda^*). \tag{3}$$

Introducing the canonical variables

$$Q_\lambda = q_\lambda + q_\lambda^*, \qquad P_\lambda = -i\nu_\lambda (q_\lambda - q_\lambda^*), \tag{4}$$

the energy of a single wave is given by

$$H_\lambda = \tfrac{1}{2}(P_\lambda^2 + \nu_\lambda^2 Q_\lambda^2). \tag{5}$$

When the radiation theory is expressed in this form, it is quite obvious how the quantum of action h is to be introduced. By exact analogy with the ordinary quantum theory we have to consider the *canonical variables of each radiation oscillator as non-commutable quantities* satisfying the *commutation relations*:

$$P_\lambda Q_\lambda - Q_\lambda P_\lambda = -i\hbar, \qquad P_\lambda Q_\mu - Q_\mu P_\lambda = 0. \tag{6}$$

The result of this quantization for the Hamiltonian (5) is given by the well-known wave-mechanical treatment of a harmonic oscillator. The eigenvalues of the energy of such an oscillator are given by

$$E_\lambda = (n_\lambda + \tfrac{1}{2})\hbar\nu_\lambda, \tag{7}$$

where n_λ is an integer. The amplitude Q_λ can be represented as a Hermitian matrix† for each λ:

$$Q_{n,n+1} = Q_{n+1,n}^* = \sqrt{\left(\frac{\hbar(n+1)}{2\nu}\right)} e^{-i\nu t} \tag{8}$$

$$Q_{n,n'} = 0 \qquad \text{if } n' \neq n \pm 1.$$

Thus Q_λ has only matrix elements for transitions in which the quantum number n_λ of the λth radiation oscillator increases or decreases by one. According to (4) the complex amplitudes q, q^* which are not Hermitian, can be represented as follows

$$\left.\begin{aligned} q_{n,n+1} &= \sqrt{\left(\frac{\hbar(n+1)}{2\nu}\right)} e^{-i\nu t} \\ q_{n+1,n}^* &= \sqrt{\left(\frac{\hbar(n+1)}{2\nu}\right)} e^{+i\nu t} \\ q_{n+1,n} &= q_{n,n+1}^* = 0. \end{aligned}\right\} \tag{9}$$

The use of the matrix (q) is advantageous, because its elements are all proportional to the *same* exponential time factor $\exp(-ivt)$,

† See any book on wave mechanics.

whereas the various matrix elements of Q have different time factors ($\exp(+i\nu t)$ and $\exp(-i\nu t)$).

According to (9) the q's satisfy the commutation relation

$$q_\lambda q_\lambda^* - q_\lambda^* q_\lambda = \frac{\hbar}{2\nu_\lambda}. \tag{10}$$

Each radiation oscillator has an energy which is an integral multiple of $\hbar\nu_\lambda$, in agreement with the original hypothesis of Planck (1). It seems, however, as though each oscillator had a *zero point energy* $\frac{1}{2}\hbar\nu_\lambda$ even in its lowest state $n_\lambda = 0$. Since the number of radiation oscillators, for given volume, is infinite, this conclusion leads us to ascribe to the vacuum an infinite zero point energy. This difficulty is, however, purely formal. This method of making the transition from the classical theory to the quantum theory is not unique, since the q, q^* are non-commutable quantities. The Hamiltonian (5) can also be written in terms of the q's

$$H_\lambda = \nu_\lambda^2 (q_\lambda q_\lambda^* + q_\lambda^* q_\lambda). \tag{11}$$

But (11) may equally well be written with the order of q_λ^* and q_λ interchanged in one of its terms without disturbing the correspondence with the classical theory. We may, for instance, write instead of (11)

$$H_\lambda = 2\nu_\lambda^2 q_\lambda^* q_\lambda = \tfrac{1}{2}(P_\lambda^2 + \nu_\lambda^2 Q_\lambda^2) - \tfrac{1}{2}\hbar\nu_\lambda. \tag{12}$$

Hence the Hamiltonian (12) has the eigenvalues

$$E_\lambda = n_\lambda \hbar\nu_\lambda \tag{13}$$

and the zero point energy has disappeared.†

A state of the radiation field is now described by the numbers n_λ for all radiation oscillators.

We consider now the momentum of the field, which is classically defined by

$$\mathbf{G} = \frac{1}{4\pi} \int [\mathbf{EH}]\, d\tau. \tag{14}$$

$\mathbf{G}$ can also be represented as a sum

$$\mathbf{G} = \sum_\lambda \mathbf{G}_\lambda, \tag{15}$$

where $\mathbf{G}_\lambda$ is the momentum of a plane wave:

$$\mathbf{G}_\lambda = \frac{1}{4\pi} \int [\mathbf{E}_\lambda \mathbf{H}_\lambda]\, d\tau.$$

† Compare L. Rosenfeld and J. Solomon, *Journ. d. Phys.* **2** (1931), 139.

According to § 6 eqq. (11), (14), and taking into account the fact that $\int [\mathbf{A}_\lambda \operatorname{curl} \mathbf{A}_\lambda]\, d\tau = 0$, and that according to (9)

$$\dot{q}_\lambda = -i\nu_\lambda q_\lambda, \qquad \dot{q}_\lambda^* = i\nu_\lambda q_\lambda^*, \tag{16}$$

we obtain

$$\mathbf{G}_\lambda = \frac{i\nu_\lambda}{4\pi c}\{q_\lambda q_\lambda^* \int [\mathbf{A}_\lambda \operatorname{curl} \mathbf{A}_\lambda^*]\, d\tau - q_\lambda^* q_\lambda \int [\mathbf{A}_\lambda^* \operatorname{curl} \mathbf{A}_\lambda]\, d\tau\}.$$

Or inserting for $\mathbf{A}_\lambda$ the value given by § 6 eq. (15) and taking into account the equation § 6 (16) we obtain

$$\mathbf{G}_\lambda = 2\nu_\lambda c \boldsymbol{\varkappa}_\lambda q_\lambda^* q_\lambda, \qquad \kappa_\lambda = \nu_\lambda / c, \tag{17}$$

where $\boldsymbol{\varkappa}_\lambda$ represents a vector with the direction of the wave and the absolute value of the reciprocal of the wave-length. In (17) again we have chosen the order of the q, q^* so that no zero point momentum occurs.

(17) is identical with the energy function (12) apart from a numerical factor. The momentum commutes therefore with the energy, its eigenvalues being

$$\mathbf{G}_\lambda = c\boldsymbol{\varkappa}_\lambda n_\lambda \hbar = n_\lambda \mathbf{k}_\lambda, \qquad |k_\lambda| = \hbar\nu_\lambda, \tag{18}$$

where $\mathbf{k}_\lambda$ is a vector with the direction of propagation and the value $\hbar\nu_\lambda$.

Thus the energy and momentum of a light wave are integral multiples of a unit $\mathbf{k}_\lambda = \hbar\nu_\lambda$. Furthermore we have seen in § 2 that they transform like a 4-vector under a Lorentz transformation. In its energy and momentum properties a plane wave behaves therefore exactly as a *beam of n free particles each with energy $\hbar\nu$ and momentum* $\mathbf{k}$ ($k = \hbar\nu$). These particles are called *light quanta*. The rest energy of a light quantum is, according to (13) and (18), equal to zero

$$G_\lambda^2 - E_\lambda^2 = 0. \tag{19}$$

We shall see later that, for the interaction of a light quantum with a free electron for instance, energy and momentum are conserved.

On the other hand, it will be seen that the quantized wave still has classical *wave properties* showing interference phenomena, etc.

The transformation properties of a light quantum under a Lorentz transformation are the same as those deduced in § 2 for the total momentum and energy of a particle (or a light wave). From § 2 eq. (44) we obtain for a Lorentz system moving in the x-direction

$$k_x' = (k_x - \beta k)\gamma, \qquad k_y' = k_y, \qquad k' = (k - \beta k_x)\gamma \tag{20}$$
$$\gamma = 1/\sqrt{(1-\beta^2)},$$

or if we denote the angle between k and x by θ and write (20) in terms of the frequencies

$$\nu' = \nu\frac{1-\beta\cos\theta}{\sqrt{(1-\beta^2)}}, \qquad \cos\theta' = \frac{\cos\theta-\beta}{1-\beta\cos\theta}. \tag{21}$$

The first equation represents the well-known formula for the *Doppler effect*. The second, which shows that the direction of the light quantum in a moving system of coordinates is different from that in the system at rest, gives the *aberration*. Both effects are of course *classical* (but relativistic) and can also easily be deduced from the transformation formulae in § 2.

For each plane wave λ we have chosen a certain direction of polarization $\mathbf{e}_\lambda$. Therefore, each light quantum also has a given polarization. According to the original choice of these directions, we have light quanta with linear, circular . . . etc., polarization.†

Finally we shall add a few remarks about the *angular momentum* of light quanta.

We have quantized the field by expanding it in a series of plane waves. We could also, of course, expand the field in many other ways. For some problems, for instance for the computation of the internal conversion coefficient or for the definition of the angular momentum of a light wave, it is more convenient to expand the field in a series of *spherical waves* with a certain point O as centre. The dependence on angle of the resulting 'eigenwaves' (i.e. of the components of the field strengths) is essentially that of the *spherical harmonics*

$$P_l^m(\cos\theta)\exp(im\phi).$$

It can be shown from the considerations of § 3 (subsection 3) that these waves are exactly those which are emitted by an electric *dipole*, *quadripole*, *magnetic dipole*, etc., situated at O. The radiation emitted by an electric dipole, for instance, has essentially an angular distribution given by the first spherical harmonic $P_1^m(\cos\theta)\exp(im\phi)$. We can therefore call these waves *dipole, quadripole* . . . radiation.‡

The amplitudes of these waves have to be quantized in exactly

† Going over from one set of two independent polarizations (for each κ) to another set, the amplitudes q_1 and q_2 of the two waves with the same frequency and direction of propagation but different polarization are transformed in a certain way. We shall not discuss this here. Cf. W. Pauli, *Handb. d. Physik*, 2nd ed. XXIV, 1.

‡ More precisely: The components of the electrical field strength of an electrical $2l$-pole are represented by two terms P_{l-1} and P_{l+1}, the magnetic field strength by a single term P_l. For a magnetic $2l$-pole the roles of E and H are changed.

the same way as for the plane waves. We obtain thus light quanta which are not represented by plane waves but by spherical waves.

It can now be shown that in this representation each 'eigenwave' has a definite *angular momentum* the z-component of which for a single light quantum is (just as for an electron without spin) given by†

$$M_z = \frac{1}{c^2}\int d\tau\,[\mathbf{RS}]_z = m\hbar, \qquad M^2 = \hbar^2 l(l+1), \tag{22}$$

where $\mathbf{R}$ represents the radius vector from O, l, m the 'quantum numbers' attached to the spherical harmonics, and $\mathbf{S}$ the Poynting vector.‡

The total angular momentum of atom+radiation is conserved in the emission of light by an atom. We shall not, however, make use of this representation of the field in this book.

The physical content of the quantum theory of transverse waves developed in this section is included essentially in Planck's assumption (1) (or, as we shall see in subsection 5, in the uncertainty relation (2)). The consequences of the theory can in fact all be derived—at least qualitatively—from these elementary laws. Our theory is simply a consistent formalism erected on the basis of Planck's original assumption (1).

The 'dual nature' of light, as a wave and as a beam of free particles, which results from this quantization is analogous to that of a beam of free electrons, which have the nature of particles and of de Broglie waves. This analogy was extraordinarily fruitful in the development of the quantum theory, but it should not now be overstressed. The existence of a discrete set of light quanta is only a result of the quantization. The corresponding classical theory is essentially a *field theory* since if, in the theory, we make $\hbar \to 0$, the light quanta have no further existence; whereas, for a beam of electrons, the wave properties are due to the quantization and the classical theory is essentially a particle theory. The particle properties of the light

† In the evaluation of the integral (22) it is not sufficient to take the field in the wave zone only (viz. the part of the field which decreases as R^{-1}). The latter gives no contribution to the integral as can be seen immediately from § 3 eq. (15). The angular momentum of a light wave is contained only in that part of the field in which E decreases as R^{-2} and H as R^{-1} i.e. in the intermediate region between wave zone and static zone. In § 3. 3 this part of the field has been neglected. In the classical theory this has been shown by M. Abraham, *Phys. Zs.* **15** (1914), 914.

‡ The question of the angular momentum of light is also discussed in Appendix II.

quanta are comprised by the above-mentioned energy and momentum relations. But there is no indication that, for instance, the idea of the 'position of a light quantum' (or the 'probability for the position') has any simple physical meaning. (Compare Landau and Peierls, loc. cit.)

3. *Statistics of light quanta.* To ensure that in the transition to the classical theory quantum electrodynamics should go over into a field theory it is essential that the light quanta should satisfy the *Einstein-Bose statistics*. This is evident, since the light quanta occur in the theory only as quantum numbers attached to the radiation oscillators. Two light quanta cannot therefore be distinguished from each other. Furthermore, the number of quanta attached to each oscillator is not limited. Considering the radiation oscillators as 'quantum cells' a state of the total radiation field is described by the number of indistinguishable particles per quantum cell. In statistical mechanics these are just the variables by which a microscopic state of an Einstein-Bose assembly is defined. Applying the usual statistical methods to obtain the state of the thermal equilibrium, our quantization leads immediately to Planck's distribution law. (Compare also § **11**. 2.)

If the light quanta satisfied the Fermi-Dirac statistics, i.e. if each radiation oscillator contained not more than one quantum, one could never obtain a field theory by making the transition to the classical theory. For if this were possible the intensity even of a radio wave could not be greater than $\hbar\nu$ and would decrease with increasing wave-length. Thus long waves could practically not exist at all. The principle of superposition, which is characteristic for the classical field theory, would not be valid, because by superposing two waves with equal wave-lengths and equal phases, we can obtain a wave with the same wave-length but higher intensity.

Thus, a classical field theory cannot exist for a Fermi-Dirac assembly; the latter can only behave classically as a system of *particles*.†

† This can also be seen formally in the following way: If one quantizes a field, ψ say, so that the particles arising from this quantization satisfy the Fermi-Dirac statistics, the amplitudes of the field at two distant points P and P' must satisfy a commutation relation (see, for instance, Heisenberg, *Physical Principles of the Quantum Theory*)

$$\psi(P)\psi(P')+\psi(P')\psi(P) \sim \hbar.$$

The de Broglie waves representing a system of electrons satisfy an equation of this form (second quantization) whereas the amplitudes of a field representing an Einstein-Bose assembly satisfy commutation relations of the type

$$\psi(P)\psi(P')-\psi(P')\psi(P) \sim \hbar$$

(see, for instance, the commutation relations of the electromagnetic field strengths

4. *The general field.*† Hitherto we have only quantized the transverse part of the field. The quantization of the longitudinal waves, however, does not lead to any new results. According to § 6 eqq. (32), (35a), (41) the longitudinal part of the field is described by two sets of canonical variables q_σ, p_σ and a_σ, b_σ which have, to satisfy the initial conditions

$$\nu_\sigma q_\sigma = -b_\sigma, \qquad \nu_\sigma p_\sigma = \ddot{a}_\sigma = -\nu_\sigma^2 a_\sigma + \sum_k e_k \phi_\sigma(k), \tag{23}$$

where $\phi_\sigma(k)$ represents the value of the Fourier component ϕ_σ of the scalar potential at the position of the kth particle. If we quantize these longitudinal waves we have to put, as in (6),

$$\begin{aligned} p_\sigma q_\sigma - q_\sigma p_\sigma &= -i\hbar, \\ b_\sigma a_\sigma - a_\sigma b_\sigma &= -i\hbar. \end{aligned} \tag{24}$$

The only question is whether the conditions (23) can also be maintained in the quantum theory. Since the q_σ, p_σ and the b_σ, a_σ are independent variables, (23) cannot be identities. They can only be satisfied in the sense that the wave function of the whole system must be chosen so that $(\nu_\sigma q_\sigma + b_\sigma)\Psi = 0$. The second condition can be satisfied in the same sense if both conditions *commute*. But this is actually the case, since from (24) one can easily see that

$$(b_\sigma + \nu_\sigma q_\sigma)(\nu_\sigma a_\sigma + p_\sigma) - (\nu_\sigma a_\sigma + p_\sigma)(b_\sigma + \nu_\sigma q_\sigma) = 0. \tag{25}$$

($\phi_\sigma(k)$ is a pure number and must therefore commute.)

The deductions of § 6 can then be carried out in exactly the same way. The longitudinal waves can be eliminated entirely and replaced by the static interaction of the particles. The quantum theoretical Hamiltonian of the total system can therefore be written (inserting eq. (13) for H_λ and omitting the external field $e_k \phi^e(k)$)

$$H = \sum_k H_k + \sum_\lambda n_\lambda \hbar \nu_\lambda + \frac{1}{2} \sum_{i \neq k} \frac{e_i e_k}{r_{ik}}, \tag{26}$$

where H_k represents the Hamiltonian of the kth particle. H_k contains the interaction with the light waves (§ 6. 4) and therefore also the effect of retardation

$$H_k = \{\mu_k^2 + [\mathbf{p}_k - e_k \sum_\lambda (q_\lambda \mathbf{A}_\lambda + q_\lambda^* \mathbf{A}_\lambda^*)]^2\}^{\frac{1}{2}}. \tag{27}$$

§ 8, eqq. (10)). Carrying out the transition to the classical theory, $\hbar \to 0$, the amplitudes commute in the case of Einstein-Bose statistics, whereas the amplitudes of a field representing a Fermi-Dirac assembly do not commute even for two very distant points and for $\hbar \to 0$. A field of the latter type has not the properties of a classical field.

† Compare E. Fermi, *Rev. Mod. Physics* (loc. cit. p. 46).

(For the quantum theoretical form of (27) see § **10**.) For q_λ, q_λ^* the matrices (9) have of course to be inserted.

In (26) again we have omitted terms e^2/r_{ii} which give an infinite self-energy of the particles. This difficulty remains unchanged in the quantum theory. In § **18** we shall even see that the quantization of the transverse waves leads to an additional diverging self-energy arising from the interaction of the particles with light.

Also the electric field strength $\mathbf{E}$ can be expressed by the amplitudes of the transverse waves and the coordinates of the particles only, in the same way as in the classical theory (§ **6** eq. (54)). According to (23) we obtain

$$\mathbf{E} = -\frac{1}{c}\sum_\lambda (\dot{q}_\lambda \mathbf{A}_\lambda + \dot{q}_\lambda^* \mathbf{A}_\lambda^*) - \frac{1}{c}\sum_\sigma \frac{1}{\nu_\sigma}\mathbf{A}_\sigma \sum_k e_k \phi_\sigma(k). \tag{28}$$

We see that the longitudinal part of the energy of the total system remains unquantized. This corresponds to the fact pointed out in § **2** that the energy and momentum of this part of the field do not represent a 4-vector and do not behave like energy and momentum in a Lorentz transformation of a particle. The longitudinal field cannot therefore be responsible for the existence of light quanta as can the transverse part of the field, which behaves like a particle in a Lorentz transformation.

5. *Light quanta, phases, and similar questions.* The quantities describing the radiation field, such as field strengths, number of light quanta, etc., no longer have definite numerical values in the quantum theory; they are now quantum-mechanical quantities which in general *do not commute.* Any two of those quantities will satisfy a certain commutation relation which will determine their behaviour. In the following section the commutation relations of the field strengths will be treated in greater detail. Here we shall consider shortly those relations in which the *number of light quanta* is involved.

The actual number of light quanta of the λth radiation oscillator can also be represented as a matrix

$$N_\lambda = n_\lambda \delta_{n_\lambda n_\lambda'}. \tag{29}$$

N_λ does not commute with the *transverse* part of the electric field strength $\mathbf{E}$, eq. (28); it commutes with the longitudinal part. For a single transverse wave we obtain, according to (9) and (29)

$$(\mathbf{E}N - N\mathbf{E})_{nn'} = (n' - n)\mathbf{E}_{nn'} = i\nu(n - n')(q_{nn'}\mathbf{A} - q_{nn'}^*\mathbf{A}^*). \tag{30}$$

Since $q_{nn'}$ is different from zero only if $n' = n+1$ and $q^*_{nn'}$ only if $n' = n-1$, we obtain

$$\mathbf{E}N - N\mathbf{E} = \pm\mathbf{E} \quad (\text{for } n' = n \pm 1). \tag{31}$$

From every quantum-mechanical commutation relation a corresponding uncertainty relation can always be deduced. If two physical quantities A and B satisfy the equation

$$AB - BA = C. \tag{32}$$

A and B satisfy the uncertainty relation

$$\Delta A \Delta B \geqslant |C|, \tag{33}$$

which has the following meaning: if the values of A and B have been determined approximately and if there is an uncertainty ΔA in our knowledge of A the uncertainty in our knowledge of B must be greater than $C/\Delta A$. Every experimental attempt to exceed the limits to our knowledge given by (33) by an exact measurement of, say, first A and then B fails because of the interaction between the measuring apparatus and the system. The accuracy of the knowledge of the value of A obtained by the first measurement is then diminished by the measurement of B to such an extent that (33) is again satisfied.

From (31) we obtain therefore the uncertainty relation

$$\Delta N \Delta E \sim E, \tag{34}$$

which means: if the electric field strength is known with an accuracy comparable with its own value the number of light quanta is uncertain by at least one unit. On the other hand, if N has a given value, the electric field strength has no definite magnitude but will fluctuate about a certain average value. This is the case even if no light quanta at all are present ($N = 0$). Although the average value of E is then equal to zero, $\mathbf{E}$ will show certain fluctuations about this value.† These zero point fluctuations of the electric field will, for instance, give rise to a certain self-energy of a free electron in empty space (see § **18**). We obtain the same fluctuations of $\mathbf{E}$ if particles are present which produce a static field only, because according to (28) the latter is entirely determined by the position of the particles.

Instead of the electric field strength $\mathbf{E}$ we can introduce the *phase* ϕ of the wave, putting

$$q = \sqrt{\left(\frac{\hbar}{2\nu}\right)} e^{i\phi} \sqrt{N}, \qquad q^* = \sqrt{\left(\frac{\hbar}{2\nu}\right)} \sqrt{N}\, e^{-i\phi}.$$

† The zero point fluctuations of $\mathbf{E}$ have no direct connexion with the zero point energy (subsection 2) which is of purely formal character.

For ϕ we then obtain the equation

$$e^{i\phi}N - Ne^{i\phi} = e^{i\phi}. \tag{35}$$

(35) is satisfied if ϕ and N satisfy the commutation relation†

$$\phi N - N\phi = -i \tag{36}$$

and the uncertainty relation

$$\Delta N \Delta\phi = 1. \tag{37}$$

Thus the number of light quanta N and the phase ϕ (multiplied by $\hbar$) are canonically conjugate. From (37) it follows that if *the number of light quanta of a wave are given, the phase of this wave is entirely undetermined* and vice versa; if the phases are known, we know nothing about the number of light quanta of the wave. If for two waves the phase difference is given (but not the absolute phase) the total number of light quanta may be determined, but it is uncertain to which wave they belong.

Finally, we can now also show that for a quantized light wave an uncertainty relation (2) is valid which was postulated in subsection 1. This uncertainty relation, however, does not refer to the position and momentum of a light quantum, since the idea of position of a light quantum has no definite meaning. Equation (2) expresses the following fact: if a beam of light is such as to give an image of a point (electron) in the x-direction, say, with an accuracy Δx, the x-component of the momentum of this light beam is uncertain by an amount $\Delta G_x = c\hbar/\Delta x$.

According to classical optics an image of a point can be formed by a monochromatic convergent beam of light with a solid angle of aperture θ, say, and a wave-length λ Owing to the diffraction, the focus has, however, a finite extension in the x-direction given by the formula

$$\Delta x = \frac{\lambda}{\sin\theta}. \tag{38}$$

(38) represents also the extension of the image, i.e. the inaccuracy of the measurement of the position of the electron.

A convergent beam of light can be obtained by superposition of plane waves with the same wave-length but various directions of propagation $\varkappa$. These plane waves have, however, to be superposed

† This can be shown in the following way: by reiterated application of (36) we can easily deduce the equation

$$\phi^n N - N\phi^n = -in\phi^{n-1}.$$

Developing the exponential function $\exp(i\phi) = \sum (i\phi)^n/n!$, (35) can be proved immediately.

with given phase differences, otherwise the beam has no definite focus. From (37) it follows then that the number of light quanta of each plane wave is not determined. Since the momentum **G** of the beam is directly given by the number of light quanta of each plane wave, **G** is not determined either.

The exact quantum-mechanical representation of such a convergent beam of light is rather complicated.† We can, however, easily obtain the uncertainty in **G** by the following consideration: supposing the total number of light quanta of the beam is equal to 1 it is uncertain to which plane wave this quantum belongs i.e. its *direction* is not determined. It can have any direction within the angle of aperture θ. The inaccuracy of G_x is therefore given by

$$\Delta G_x = k \sin\theta = \hbar\nu \sin\theta. \tag{39}$$

From (38) and (39) we obtain, in fact, the uncertainty relation

$$\Delta G_x \Delta x = \hbar c \tag{40}$$

which was postulated in subsection 1.

We see that the uncertainty relation (40) is a simple consequence of the quantization, i.e. of the fact that for a given frequency the momentum cannot be smaller than $\hbar\nu$.

8. Commutation and uncertainty relations of the field strengths

1. *Commutation relations of the field strengths.* The quantization of the electromagnetic field that was carried out in § 7 showed the field strengths to be no longer simple functions of space and time but quantum-mechanical quantities which in general *do not commute.*

The quantization was made possible by the development of the potentials in a series of monochromatic plane waves (§ 7 eq. (3), § 6 (32), (35 a))

$$\mathbf{A}(\mathbf{r}, t) = \sum_\lambda (q_\lambda \mathbf{A}_\lambda + q_\lambda^* \mathbf{A}_\lambda^*) + \sum_\sigma q_\sigma \mathbf{A}_\sigma \tag{1}$$

$$\phi = \sum_\sigma a_\sigma \phi_\sigma,$$

where the $\mathbf{A}_\lambda$, $\mathbf{A}_\sigma$, ϕ_σ are ordinary functions of space, whereas the q_λ, q_σ are quantum-mechanical quantities (matrices, q-numbers) which satisfy certain commutation relations of which examples are

$$[q_\lambda q_\lambda^*] = q_\lambda q_\lambda^* - q_\lambda^* q_\lambda = \frac{\hbar}{2\nu_\lambda}, \qquad [q_\lambda q_\mu^*] = 0. \tag{2}$$

† Compare K. F. v. Weizsäcker, *Zs. f. Phys.* **70** (1931), 114.

The matrix q_λ is proportional to $\exp(-i\nu_\lambda t)$ and q_λ^* to $\exp(+i\nu_\lambda t)$. Therefore the potentials (1) and the *field strengths, at any given point* $\mathbf{r}$ *of space and at any time* t, *are non-commutative quantities.*

In quantum electrodynamics the commutation relations satisfied by the field strengths are as essential to the theory as are the well-known relations for the position and momentum of an electron in quantum mechanics. The relations for the field quantities, however, are much more complicated, since for two components of the field strengths they will depend upon the two points in space and instants of time at which the field strengths are considered.

These commutation relations can be deduced from equations (1) and (2). We consider, for instance, two components of the magnetic field strength $H_i(\mathbf{r}_1, t_1)$ and $H_k(\mathbf{r}_2, t_2)$ at two points in space time $\mathbf{r}_1$, t_1 and $\mathbf{r}_2$, t_2. Since $\operatorname{curl} \mathbf{A}_\sigma = 0$ we obtain from (1) inserting for $\mathbf{A}_\lambda$ the function § 6 eq. (15a):

$$\left.\begin{aligned} H_i(\mathbf{r},t) = i\sqrt{(4\pi c^2)} \sum_\lambda [\boldsymbol{\kappa}_\lambda \mathbf{e}_\lambda]_i \{q_\lambda e^{i(\boldsymbol{\kappa}_\lambda \mathbf{r} - \nu_\lambda t)} - q_\lambda^* e^{-i(\boldsymbol{\kappa}_\lambda \mathbf{r} - \nu_\lambda t)}\} \\ (i = x, y, z), \end{aligned}\right\} \tag{3}$$

where $\mathbf{e}$ is the unit vector in the direction of polarization.

In (3) we have separated from q_λ the time factor $\exp(-i\nu_\lambda t)$. Denoting the point $\mathbf{r}, t$ simply by P, we obtain the commutation relations, according to (2),

$$\begin{aligned} [H_i(P_1)H_k(P_2)] &\equiv H_i(P_1)H_k(P_2) - H_k(P_2)H_i(P_1) \\ &= -i4\pi c^2\hbar \sum_\lambda [\boldsymbol{\kappa}_\lambda \mathbf{e}_\lambda]_i [\boldsymbol{\kappa}_\lambda \mathbf{e}_\lambda]_k \frac{1}{\nu_\lambda} \sin[(\boldsymbol{\kappa}_\lambda, \mathbf{r}_2 - \mathbf{r}_1) - \nu_\lambda(t_2 - t_1)]. \end{aligned} \tag{4}$$

If we carry out the summation over the directions of polarization $\mathbf{e}_\lambda$ ($\mathbf{e}_\lambda$ is always $\perp \boldsymbol{\kappa}_\lambda$) and write $\mathbf{r}$ for $\mathbf{r}_2 - \mathbf{r}_1$ and t for $t_2 - t_1$, we obtain

$$\begin{aligned} [H_i(P_1)H_k(P_2)] &= -4\pi i\hbar c \sum (\kappa^2 \delta_{ik} - \kappa_i \kappa_k) \frac{\sin[(\boldsymbol{\kappa}\mathbf{r}) - \kappa ct]}{\kappa} \\ &= -4\pi i\hbar c \left(\frac{1}{c^2} \frac{\partial^2}{\partial t_1 \partial t_2} \delta_{ik} - \frac{\partial^2}{\partial x_{i1} \partial x_{k2}}\right) \sum \frac{\sin[(\boldsymbol{\kappa}\mathbf{r}) - \kappa ct]}{\kappa}. \end{aligned} \tag{5}$$

Here the summation $\sum$ has to be carried out over all directions and values of $\boldsymbol{\kappa}$. This summation can be replaced by an integration, taking into account the fact that according to § 6 (21) the number of waves per unit volume for which $\boldsymbol{\kappa}$ lies in the element $d\kappa_x\, d\kappa_y\, d\kappa_z$ is equal to

$$\frac{1}{(2\pi)^3} d\kappa_x\, d\kappa_y\, d\kappa_z. \tag{5'}$$

Integrating first over all directions of $\boldsymbol{\varkappa}$ and then over $\varkappa$ we have

$$\int\int\int \frac{d\varkappa_x\, d\varkappa_y\, d\varkappa_z}{\varkappa} \sin[(\boldsymbol{\varkappa}\mathbf{r}) - \varkappa ct] = \frac{2\pi}{r} \lim_{\varkappa\to\infty} \left[\frac{\sin \varkappa(r+ct)}{r+ct} - \frac{\sin \varkappa(r-ct)}{r-ct} \right]. \quad (6)$$

The function on the right-hand side of (6) is the relativistic analogue of Dirac's well-known δ-function. We denote it by $\Delta(r, t)$.

$$\Delta = \lim_{\varkappa\to\infty} \frac{1}{\pi r} \left[\frac{\sin \varkappa(r+ct)}{r+ct} - \frac{\sin \varkappa(r-ct)}{r-ct} \right]. \quad (7)$$

Dirac's δ-function $\delta(x)$ is zero everywhere except at $x = 0$. It is defined by

$$\int f(x)\, \delta(x) dx = f(0),$$

where $f(x)$ is an arbitrary function and the integration is over a region containing the point 0. The Δ function defined by (7) has very similar properties. It is zero everywhere except for those points of space time for which either $r = +ct$ or $r = -ct$.

If we integrate the product of Δ and an arbitrary function $f(\mathbf{r}, t)$ over the four-dimensional space we obtain

$$\int f(r,t)\Delta(r,t)\, d\tau dt = \int d\tau \, \frac{1}{r} [f(r = -ct) - f(r = +ct)]. \quad (8)$$

Δ can also be expressed by the ordinary δ-function:

$$\Delta = \frac{\delta(r+ct) - \delta(r-ct)}{r}. \quad (9)$$

The points at which Δ is different from zero form a double cone in four-dimensional space. These points can be reached by a light signal emitted at the origin $r = 0$ at $t = 0$ $(r = +ct)$ and conversely a light signal emitted at the point $\mathbf{r}, t$ can reach the origin $r = 0$ at the time $t = 0$ $(r = -ct)$.

The Δ function occurring in (6) contains as argument

$$|\mathbf{r}_2 - \mathbf{r}_1| - c(t_2 - t_1) \quad \text{and} \quad |\mathbf{r}_2 - \mathbf{r}_1| + c(t_2 - t_1).$$

It is therefore different from zero only if the two points of space time at which the field strengths are considered can be connected by a light signal. *Therefore the field strengths at two points of space time which cannot be connected by light signals commute with each other.*

Introducing this Δ-function in the commutation relation (5) we obtain, according to (5), (5′), (6), (7),

$$[H_i(P_1) H_k(P_2)] = -i\hbar c \left(\frac{1}{c^2} \frac{\partial^2}{\partial t_1 \partial t_2} \delta_{ik} - \frac{\partial^2}{\partial x_{i1} \partial x_{k2}} \right) \Delta. \quad (10\,a)$$

In the same way one can find the commutation relations for the other field strengths:

$$[E_i(P_1)E_k(P_2)] = [H_i(P_1)H_k(P_2)] \tag{10b}$$

$$[E_i(P_1)H_i(P_2)] = 0 \tag{10c}$$

$$[E_i(P_1)H_k(P_2)] = +i\hbar\frac{\partial^2}{\partial x_l \partial t_1}\Delta \tag{10d}$$

($i \neq k$; i, k, l forming an even permutation of x, y, z).

The relations (10) were first deduced by Jordan and Pauli.†

The longitudinal part of the field gives no contribution to the commutation relations (10), because the field strengths can be expressed as functions of the position of the particles and the transverse waves only (see § 7 eq. (28)). The longitudinal part of the field therefore commutes. It would be wrong, however, to assume that in the absence of light quanta the electric field strengths at two different points commute. In § 7. 5 we have pointed out that even if no light quanta are present the electric field strength (due to the transverse waves) does not vanish but fluctuates about the average value zero. These zero point fluctuations have to be superposed on the static part of the field, with the result that E at two different points does not commute but satisfies the relation (10).

The universal constants occurring in the commutation relations (10) are c and $\hbar$. No constant referring to the atomic structure of matter (m or e) occurs in (10). The non-commutability of the field strength which can be considered as characteristic for present quantum electrodynamics is therefore purely an effect of the union of the quantum theory with classical electrodynamics and has no connexion with the problem of the elementary particles.

2. *Uncertainty relations for the field strengths.* From the quantum-mechanical commutation relations (10) we can obtain corresponding uncertainty relations. If two physical quantities A and B satisfy the equation

$$AB - BA = C,$$

where C represents an ordinary number (not a matrix), A and B satisfy the uncertainty relation

$$\Delta A \Delta B \sim |C|. \tag{11}$$

The uncertainty relations deduced in this way from (10) have, however, no immediate physical significance. They represent rela-

† P. Jordan and W. Pauli, *Zs. f. Phys.* **47** (1928), 151.

tions for the field strengths at given points of space and time. But the only quantities which can be measured are the *average values* of the field strengths over certain *regions* of space and time. To obtain relations for these average values we integrate the equations (10) over two regions of space and time $L_1^3 T_1$ and $L_2^3 T_2$, respectively for the two field strengths occurring in each equation (10). We shall denote these two regions by I_1 and I_2 and the average values of the field strengths by E_{xI_1} or $E_{xL_1T_1}$, etc. The result of the integration on the right-hand side will depend upon the relative position of these two regions, i.e. on which point of I_1 can be reached from I_2 by a light signal and vice versa. We shall, however, confine ourselves to a few simple but characteristic cases.

(*a*) Both time regions are identical $T_1 = T_2$. According to (7) Δ is antisymmetrical in the two times t_1 and t_2. (10 a) is symmetrical in the derivatives with respect to the two times. The time integral of the right-hand side of (10 a) over $T_1 = T_2$ therefore vanishes. Thus, according to (9), (10 a), (10 b), (11),

$$\Delta E_{iL_1T} \Delta E_{kL_2T} = \Delta H_{iL_1T} \Delta H_{kL_2T} = 0. \tag{12}$$

The average values of two components of the electric or magnetic field strength over the same time region but different space regions commute and can therefore be measured exactly.

(*b*) Both space regions are identical $L_1 = L_2$. Then the integral over the right-hand side of (10 d) vanishes and we have

$$\Delta E_{iLT_1} \Delta H_{kLT_2} = 0. \tag{13}$$

The average values of a component of the electric field strength and a component of the magnetic field strength over the same space region but different time regions commute and can therefore be measured exactly.

From (12) and (13) it follows, of course, that the average values of any two components of the field strengths over the same region of *space and time* can always be measured exactly.

(*c*) The two regions I_1 and I_2 are situated so that light signals emitted from some at least of the points of I_1 can reach I_2, but no light signal emitted from I_2 can reach any point of L_1^3 during the time T_1 (Fig. 3). Then all contributions arising from the first term of the Δ-function (9) vanish.

We consider the two cases of a simultaneous† measurement of the x-component of the electric field strength E_x in I_1 and I_2 and of E_x in I_1 and H_y in I_2. The equations (10) give immediately the uncertainty relations

$$\Delta E_{xI_1}\Delta E_{xI_2} = \frac{\hbar c}{L_1^3 L_2^3 T_1 T_2}\int_{L_1 L_2 T_1 T_2}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t_1 \partial t_2}-\frac{\partial^2}{\partial x_1 \partial x_2}\right)\frac{\delta[|\mathbf{r}_2-\mathbf{r}_1|-c(t_2-t_1)]}{|\mathbf{r}_2-\mathbf{r}_1|} \quad (14)$$

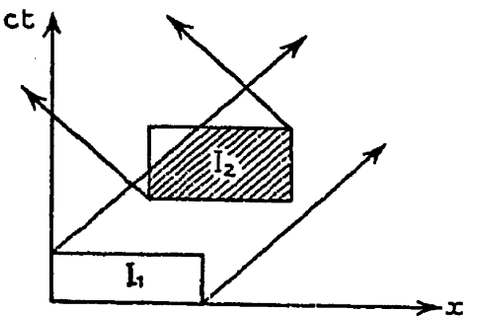

FIG. 3. Two regions of space and time. Light signals from I_1 can reach I_2, but no light signal from I_2 can reach I_1.

and

$$\Delta E_{xI_1}\Delta H_{yI_2} = \frac{\hbar}{L_1^3 L_2^3 T_1 T_2}\int_{L_1 L_2 T_2}\frac{\partial}{\partial z_2}\frac{\delta[|\mathbf{r}_2-\mathbf{r}_1|-c(t_2-t_1)]}{|\mathbf{r}_2-\mathbf{r}_1|}\Bigg|_{t_{10}}^{t_1'}. \quad (15)$$

In (15) we have carried out the integration over T_1; t_{10} and t_1' represent the times of the beginning and the end of the interval T_1 respectively.

The right-hand sides of (14) and (15) can be interpreted physically in a simple way. This will be done in subsections 3 and 4. Here we shall give only an estimate of the order of magnitude. We assume that $L_1 \sim L_2$, $T_1 \sim T_2$ and that the order of magnitude of the distance between the two space regions L_1 and L_2 is r.

Furthermore the region of I_2 which can be reached from I_1 by light signals shall be of the same order as I_2 itself (Fig. 3). The order of magnitude of the right-hand side of (15), for instance, depends on whether $L \gtrless cT$. One can easily find for the two cases

$$\Delta E_{xI_1}\Delta H_{yI_1} \sim \begin{cases} \hbar/r^2 LT & (L \gg cT) \\ \hbar/r^2 cT^2 & (L \ll cT), \end{cases} \quad (16)$$

and a similar expression for (14). Thus the two field strengths can be

† The expression 'simultaneous measurement of two quantities' is obviously not used here in the sense 'measurement at the same time', but means that the reciprocal influence of the two measurements is taken into account.

measured the more exactly the larger is the distance between the two space regions, which is a very natural result. The effect of non-commutability of the field strengths is only appreciable for neighbouring space points.

(16) gives also the condition for the quantum properties of the field to be essential or for the classical theory to be applicable. The latter is the case if the field strengths are large compared with the term expressing the effect of non-commutability given by the right-hand side of (16). For field strengths of the order E we obtain (putting the distance of the two space regions of the order L)

$$E^2L^3cT \gg \hbar c \quad (L > cT). \tag{17}$$

Thus the typical quantum region is that of *weak fields*. For a light wave of frequency ν (17) simply expresses the condition that the number of light quanta n contained in L^3 must be large: Since $E^2L^3 = n\hbar\nu$ and since the time-interval T must be chosen to be smaller than $1/\nu$ (otherwise the average value of E vanishes), we obtain from (17): $n \gg 1$.

3. *Measurement of the average value of a field strength.* For a critical understanding of the quantum-mechanical formalism it is decidedly important to make sure that the uncertainty relations to which the formalism leads are consistent with the accuracy which can be best attained by means of *measurements*. The well-known discussion of the uncertainty of the position and momentum of an electron,† for instance, shows that it is impossible to exceed the limits of accuracy given by the uncertainty relation $\Delta p \Delta q \sim \hbar$ by a direct measurement of the position of the electron (when the momentum was known beforehand). As a counterpart to this ideal experiment we shall show that our uncertainty relations for the field strengths (14), (15) are in a similar way consistent with the accuracy which can be attained by a simultaneous measurement of two field strengths.‡ For this purpose, however, we must first consider the way in which a single field strength can be measured. The quantum-mechanical formalism presumes that a single physical quantity such as the average value of the x-component of the electric field strength E_{xI} can be measured exactly. This assumption has of course first to be tested

† Cf. Heisenberg, *Die physikalischen Principien der Quantentheorie*, 15, Leipzig 1930; or Bohr, *Naturw.* (1928), 245, (1930), 73.

‡ We follow in these considerations the paper of Bohr and Rosenfeld, *Det. Kgl. dansk. Vid. Selskab.* XII (1933), 8.

before we can interpret the uncertainty relations for two field strengths.

The simplest way to measure E_{xI} would be to take a *charged test body* of mass M covering the region L^3 and having a uniformly distributed charge ϵ. If we measure the momentum p_x† of this test body at the beginning t_0 and at the end t' of the time-interval T, the average value of E_x is given by‡

$$E_{xI} = \frac{p_{x0} - p'_x}{\epsilon T}. \tag{18}$$

Since the right-hand sides of the uncertainty relations (14), (15) depend only upon the universal constants $\hbar$, c and upon the geometrical circumstances but not upon any quantity which refers to an *elementary particle* (e or m), the problem of the measurement of field strengths cannot have anything to do with the *atomic structure of matter*. Thus the test body may have any size (i.e. the region L^3 may have any size) and any charge, and we shall see in fact that the highest possible accuracy is reached with a *heavy* test body containing a large number of elementary charges. Therefore, the difficulties which are connected with elementary particles (infinite self-energy, etc.) do not play any role in our problem.

We have, however, to take into account that even a heavy test body has to satisfy the general laws of quantum mechanics especially the uncertainty relation for its position x and momentum p_x

$$\Delta x \Delta p_x = \hbar. \tag{19}$$

Therefore, if the momentum of the test body is measured at the beginning of the time-interval t_0 with an accuracy Δp_x, we know the position of the test body during the whole time T only with an accuracy $\hbar/\Delta p_x$.

Furthermore, if we measure the momentum at the time t_0 within a short interval Δt_0 ($\Delta t_0 \ll T$) a certain velocity v_{x0} will be transferred to the test body according to the well-known relation

$$v_{x0} \Delta p_{x0} \Delta t_0 = \hbar. \tag{20}$$

We shall assume that the test body is before the measurement of

† In this section we use the usual notation for 'momentum'. (Dimension gm. cm. sec.$^{-1}$.)

‡ In this procedure for carrying out the measurement it is of course assumed that the test body filling the region is a *rigid body*. It has been proved in the paper of Bohr and Rosenfeld (loc. cit.) that for this case the assumption that such rigid bodies actually exist is fully justified.

p_{x0} in a fixed position covering the region L^3 exactly. v_{x0} is then just the velocity which is necessary to displace the test body by Δx within the time Δt_0

$$v_{x0} = \frac{\hbar}{\Delta p_{x0}\,\Delta t_0} = \frac{\Delta x}{\Delta t_0}. \tag{21}$$

In contrast to the uncertainties Δx, Δp_x, the velocity v_{x0} is a *known* quantity (disregarding an uncertainty Δv_{x0} which is small if the mass M of the test body is large). In a similar way we measure the momentum after the time-interval T at t' within a short interval $\Delta t'$ and then bring the test body back to its original position covering again the region L^3 exactly.

These uncertainties (19), (20) will, as we shall see in subsection 4, give rise to the limitation of the accuracy of the simultaneous measurement of two field strengths (as given by equations (14), (15)), but they do not restrict the accuracy of the measurement of a single field strength. In this case all these uncertainties can be compensated.

The accuracy of the measurement of E_{xI} would first of all be restricted by the following facts:

(a) the inaccuracy Δp_x with which p_x at the beginning and the end of the time-interval T is measured;

(b) by the fact that the test body does not cover the region L^3 exactly during the time T because of:

- (α) the acceleration exerted by the field itself,
- (β) the velocity v_{x0} which the test body has after t_0. v_{x0} even becomes large if Δt and Δp_x are small (according to (21)),
- (γ) the unknown displacement Δx of the test body which is connected with the knowledge of p_x.

These inaccuracies can be compensated in the following way:

(b, α) The field does not change the position of the test body appreciably if the latter is sufficiently *heavy*.

(b, β) Since the velocity v_{x0} transferred to the test body by the measurement of the momentum is known (apart from a negligible uncertainty Δv_{x0}) we can compensate this effect by giving the test body a kick immediately after the measurement of the momentum, i.e. practically at the beginning of the time-interval T. The same has to be done after the second measurement of the momentum p'_x.

(b, γ) and (a). The unknown displacement Δx can only be made

small if the measurement of p_x is rather inaccurate. This does not, however, restrict the accuracy of the measurement of E_{xI} since we have another parameter available—the charge ϵ. For a small Δx the inaccuracy of the field strength is, according to (18), given by

$$\Delta E_{xI} = \frac{\Delta p_x}{\epsilon T} = \frac{\hbar}{\epsilon \Delta x T}. \tag{22}$$

Thus for any Δx, however small, E_{xI} can be measured with any desired accuracy if the *charge* ϵ of the test body is *sufficiently high*. If we want an accuracy high enough to test the quantum properties of the field we see from our estimation (16) and from (22) that ϵ has to be large compared with the elementary charge e.

But, if ϵ is large, another difficulty arises: the field which is measured accurately in the way described above is not only the external field which we wish to measure but also the field $\mathcal{E}$ produced by the test body itself, and it is just for a large charge ϵ that $\mathcal{E}$ becomes large also. This difficulty exists, however, only in so far as we do not know (and cannot calculate) this field exactly because we do not know the position and motion of the test body exactly according to (19), (20).

As we shall see in the next section, it is just this inaccuracy $\Delta\mathcal{E}$ of the field produced by the test body which gives rise to the limitation of accuracy for the measurement of two field strengths (according to (14), (15)). For the measurement of a single field strength the effect arising from the inaccuracy $\Delta\mathcal{E}$ can again be compensated.

We shall have to work out the field $\mathcal{E}$ produced by the test body in subsection 4. We see from the calculations (25)–(31) that the average value of $\mathcal{E}$ over the region L^3 is proportional to the displacement x

$$\mathcal{E}_{xI} = Fx, \qquad \Delta\mathcal{E}_{xI} = F\Delta x. \tag{23}$$

But then the force which the field $\mathcal{E}$ exerts on the test body itself can be compensated entirely if we subject the test body during the whole time-interval T to another force of a purely mechanical nature which is also proportional to the displacement x from the original position L^3. Such a force can, for instance, be realized by a spring. If we choose the strength of the spring so that the force is equal to

$$K_x = -\epsilon F x, \tag{24}$$

the force exerted by the field $\mathcal{E}$ is just compensated, whatever the—known or unknown—displacement of the test body is. The measurement of $\mathcal{E}$ carried out in this way indicates then the external field

only. For the accuracy of this measurement there exists in principle no restriction.†

4. *Measurement of two field strengths.* Our next task is the physical interpretation of the uncertainty relations (14), (15). According to the general quantum-mechanical interpretation of the uncertainty relations, we have to show that a measurement of the average value of E_x, say, over the space time region I_1 does not allow an accurate measurement of another field strength in the region I_2. The limitation of accuracy for the latter is due to the fact that the test body used for the measurement in I_1 produces in I_2 a field $\mathcal{E}$, $\mathcal{H}$, which is to some extent unknown. This unknown field is superposed on the field E, H in I_2 which we wish to measure and cannot be separated from it. It gives rise to an inaccuracy of the value of E, H in I_2 even if we measure the field in I_2 exactly.

To calculate this inaccuracy we have to work out the uncertainty of the field which is produced by the first test body. The sources of this field are the following:

(*a*) As we have seen in subsection 3, the test body during the whole time T is displaced by an unknown quantity Δx. This gives rise to the field of an *electric dipole* with the moment $\epsilon\Delta x$ in the x-direction. This dipole moment is distributed uniformly over the space L_1^3 with a density $\epsilon\Delta x/L_1^3$. The uncertainty of the scalar potential at a point $\mathbf{r}_2$, t_2, produced by a volume element $d\tau_1$ at the time t_1, is therefore, if we take into account that the field travels with the velocity c,

$$\Delta\phi(\mathbf{r}_2, t_2)\, d\tau_1 = \frac{\epsilon\Delta x}{L_1^3}\frac{\partial}{\partial x_1}\frac{c\delta(r-ct)}{r}\, d\tau_1, \tag{25}$$

where we have denoted $|\mathbf{r}_2-\mathbf{r}_1|$ by r and t_2-t_1 by t. (The factor c in (25) comes from the definition of the δ-function $c\int\delta\, dt = 1$.)

(*b*) At the beginning t_0 of the time-interval T the test body has for a short time Δt_0 a velocity v_{x0}. Thus the test body represents a *current density*‡

$$i_x d\tau_1 = \frac{\epsilon}{L_1^3} d\tau_1 \frac{\Delta x}{\Delta t_{10}} \quad (\text{at } t = t_{10}). \tag{26}$$

† We have not mentioned two points which have to be taken into account for the measurement of E_x. (1) The *reaction* of the field $\mathcal{E}_x$ produced by the test body on the test body itself; (2) the fact that the field $\mathcal{E}_x$ which we have computed above classically is also quantized. For the discussion of both points we must refer to the paper of Bohr and Rosenfeld. Neither of them actually gives rise to any restriction on the accuracy of field measurements.

‡ The formulae (25), (26) hold obviously also for a *known* displacement x as was assumed in (23).

At the time t_1' (the end of the interval T) the test body has again a velocity $v_x' = \Delta x/\Delta t_1'$. But since we bring the test body *back* to its original position L_1^3 the time integral of the current density at t_1' has the same value but the opposite sign to (26). Assuming that Δt_{10}, $\Delta t_1'$ are infinitely small we can write for the total current density

$$i_x d\tau_1 = \frac{\epsilon \Delta x \, d\tau_1}{L_1^3}[\delta(t-t_{10})-\delta(t-t')]. \tag{27}$$

This current density gives rise to a vector potential $\mathcal{A}$ at the space time point $\mathbf{r}_2$, t_2. Taking into account the retardation we have to insert $t_2 - |\mathbf{r}_2-\mathbf{r}_1|/c$ for t in (27). Thus the unknown vector potential is equal to

$$\begin{aligned}\Delta\mathcal{A}_x(\mathbf{r}_2 t_2)\, d\tau_1 &= \frac{\epsilon\Delta x}{cL_1^3} d\tau_1 c \frac{\delta[c(t_2-t_{10})-r]-\delta[c(t_2-t_1')-r]}{r} \\ &= -\frac{\epsilon\Delta x}{L_1^3} d\tau_1 \frac{\delta(r-ct)}{r}\Bigg|_{t_{10}}^{t_1'}. \end{aligned} \tag{28}$$

From the potentials (25), (28) we obtain the uncertainties of the field strengths

$$\begin{aligned}\Delta\mathcal{E}_x &= -\frac{\partial\Delta\phi}{\partial x_2} - \frac{1}{c}\frac{\partial\Delta\mathcal{A}_x}{\partial t_2} \\ \Delta\mathcal{H}_y &= \frac{\partial\Delta\mathcal{A}_x}{\partial z_2}. \end{aligned} \tag{29}$$

Integrating over all points of I_1 and taking the average over I_2, we obtain the total unknown contribution to the field in I_2 arising from the test body in I_1

$$\Delta\mathcal{E}_{xI_2} = \frac{\epsilon\Delta x c}{L_1^3 L_2^3 T_2}\int_{L_1 L_2 T_2}\left\{\frac{1}{c^2}\frac{\partial}{\partial t_2}\frac{\delta(r-ct)}{r}\Bigg|_{t_{10}}^{t_1'} - \frac{\partial^2}{\partial x_1\,\partial x_2}\int_{T_1}\frac{\delta(r-ct)}{r}\right\} \tag{30}$$

$$\Delta\mathcal{H}_{yI_2} = \frac{\epsilon\Delta x}{L_1^3 L_2^3 T_2}\int_{L_1 L_2 T_2}\frac{\partial}{\partial z_2}\frac{\delta(r-ct)}{r}\Bigg|_{t_{10}}^{t_1'}. \tag{31}$$

The formulae (30), (31) give the inaccuracy of the field measurement in I_2. We can therefore write

$$\Delta\mathcal{E}_{xI_2} = \Delta E_{xI_2}.$$

Since we have assumed in subsection 2 that no light signal emitted from I_2 can reach the region I_1, no similar disturbance of the measurement in I_2 by the test body in I_1 can exist.

If we multiply (30), (31) by the inaccuracy of the field measurement in I_1 as given by (22)

$$\Delta E_{xI_1} = \frac{\hbar}{\epsilon \Delta x T_1}, \tag{32}$$

we obtain exactly the *uncertainty relations* (14), (15), deduced from the formalism. The charge ϵ and the unknown displacement Δx cancel of course.

If the two regions are situated so that also light signals from I_2 can reach I_1, the arrangement of the measurements which is necessary to reach the highest possible accuracy is more complicated. This case has been treated in the paper of Bohr and Rosenfeld (loc. cit.), where also all other details concerning the problem of field measurements have been carefully discussed.

Thus we have proved that the consequences of the quantum electrodynamical formalism agree with the possibilities given by field measurements. This proof is completely analogous with the proof of the uncertainty relation for the position and momentum of an electron in quantum mechanics. In the latter the quantum properties of a *light beam* in the γ-ray microscope prevent an exact measurement of the *position and momentum* of the electron (§ 7. 5). In our case the quantum *mechanical* properties of the test body prevent an exact measurement of two *field strengths*. From this point of view it becomes clear that quantum electrodynamics and quantum mechanics form two inseparable parts of a single body of the quantum theory, that both are logically connected and neither is consistent without the other. The quantum theory of the field can therefore be considered as the reasonable quantum theoretical extension of classical electrodynamics in the same way as quantum mechanics is the quantum theoretical extension of classical mechanics.

III

INTERACTION OF RADIATION WITH MATTER

9. Quantum-mechanical introduction

BEFORE applying the quantum theory of the electromagnetic field to the problem of the interaction of radiation with matter it will be convenient to summarize some of the results of the quantum theory which will frequently be used in our subsequent discussion. In subsections 1 and 2 of the following section these results will be quoted without proof.

1. *Non-relativistic wave equation.* The motion of an electron in a given field in the non-relativistic case is described, according to § 6 eq. (26), by the Hamiltonian

$$H = e\phi + (\mathbf{p} - e\mathbf{A})^2/2\mu = E, \tag{1}$$

where $\mathbf{p}$ represents the momentum (in energy units, i.e. c multiplied by the impulse), E the total energy, and μ the rest energy of the electron. In quantum theory p_x and E are replaced by

$$p_x = -i\hbar c\frac{\partial}{\partial x}, \qquad E = +i\hbar\frac{\partial}{\partial t}. \tag{2}$$

The wave equation belonging to the Hamiltonian (1)

$$H\psi = E\psi$$

has the following solutions:

(*a*) for a *free electron* $\phi = \mathbf{A} = 0$

$$\psi_{\mathbf{p}} = e^{i(\mathbf{pr})/\hbar c}e^{-iEt/\hbar}. \tag{3}$$

(3) represents a plane wave with the direction of propagation $\mathbf{p}$. The wave function is normalized for a *unit volume*, i.e. so that

$$\int_{V=1} |\psi|^2\, d\tau = 1.$$

Assuming the wave-length to be small compared with the dimensions of a cube of unit volume, the number of stationary states for which the direction of the momentum lies within the element of solid angle $d\Omega$ and its magnitude p between p and $p+dp$ is given by

$$\rho_E dE d\Omega = \frac{dp_x dp_y dp_z}{(2\pi\hbar c)^3} = \frac{p^2\, dp d\Omega}{(2\pi\hbar c)^3} = \frac{pE\, dE d\Omega}{(2\pi\hbar c)^3}. \tag{4}$$

This equation also holds in the relativistic case where $E^2 = p^2 + \mu^2$.

For the non-relativistic case one has to insert in the last equation (4) $E = \mu$, $dE = p\,dp/\mu$. (4) gives the number of states with a *given spin direction*. The total number of states is twice as large.

(*b*) For the *Coulomb field* of a nucleus with charge Ze,

$$V = -Ze^2/r,$$

the stationary states are given by the Balmer formula. It is convenient to express the ionization energy I of the ground state in units of the rest energy μ although, of course, I is independent of the velocity of light:

$$I = \frac{Z^2e^4m}{2\hbar^2} = \frac{Z^2e^2}{2a_0} = \frac{1}{2}\left(\frac{Z}{137}\right)^2\mu, \tag{5}$$

where

$$\left\lq\frac{1}{137}\right\rq = \frac{e^2}{\hbar c} = \frac{1}{137{\cdot}3} \tag{6}$$

is the fine structure constant and a_0 Bohr's radius of the H atom ($Z = 1$). a_0 is connected with the other universal lengths $\lambda_0 = \hbar/mc$ (Compton wave-length) and $r_0 = e^2/mc^2$ (classical electronic radius) by the formulae

$$a_0 = \frac{\hbar^2}{me^2} = \lambda_0 \times 137 = r_0 \times 137^2. \tag{7}$$

The normalized wave function of the ground state is given by

$$\psi = \frac{1}{\sqrt{(\pi a^3)}}\,e^{-r/a}, \quad a = \frac{a_0}{Z}. \tag{8}$$

a is the radius of the K-shell.

The wave functions for the *continuous spectrum* are more complicated.† We limit ourselves to a few remarks. If the energy of the electron is large compared with the ionization energy of the K-electron, the wave function can be replaced by the wave function of a free electron (3). This is the basis of *Born's method* for the treatment of collision problems.‡ The exact criterion for the validity

† Cf. Mott and Massey, *Theory of Atomic Collisions*, Oxford University Press, 1933, p. 34.

‡ This remark has to be understood in the following way: the use of Born's approximation means that the interaction V between the electron and the nucleus (or other particles) is considered as a small perturbation, and only the first approximation in this interaction which does not vanish is taken into account. Considering the scattering of an electron in the field of the nucleus, we can insert plane waves for the wave function of the electron in the matrix element of V. If, however, we consider a higher process, for instance the radiation emitted by the deflexion of an electron in the field of the nucleus (§ 17), we have to insert, in the matrix element of the interaction H with the radiation, the next highest approximation for the wave function of

of this approximation is

$$2\pi\xi = 2\pi\frac{Ze^2}{\hbar v} \equiv \frac{2\pi Z}{137\beta} \ll 1, \quad \beta = \frac{v}{c}, \tag{9}$$

where v is the velocity of the electron at great distances from the nucleus. This condition holds also in the relativistic case where it is always satisfied ($v \sim c$) except for the heaviest elements (even then $Z/137\beta < 1$). If (9) is not satisfied, i.e. if the energy of the electron is of the same order of magnitude as the ionization energy, the wave function which behaves at great distances from the nucleus as a plane wave normalized as in equation (3) travelling in the x-direction with the momentum p, can, in the neighbourhood of the nucleus, be represented as a series:

$$\psi = e^{ipx/\hbar c}e^{\pi\xi/2}\Gamma(1-i\xi)\left[1-\frac{\xi p}{\hbar c}(r-x)-\ldots\right], \tag{10}$$

where Γ is the gamma function and r the distance from the nucleus. At the position of the nucleus ($r = x = 0$) $|\psi|^2$ is given simply by

$$|\psi|^2 = |\Gamma(1-i\xi)|^2 e^{\pi\xi} = \frac{2\pi\xi}{1-e^{-2\pi\xi}}. \tag{11}$$

The formulae (9), (10), (11), hold for positive as well as negative electrons (see Chapter IV); for a positive electron the sign of ξ has to be reversed.

In the neighbourhood of the nucleus $|\psi|^2$ is *larger* than at great distances for a negative electron and smaller for a positive one. For $2\pi\xi \ll 1$ (condition for the validity of Born's approximation) (10) reduces to a plane wave.

2. *Relativistic wave equation.* In the relativistic case the Hamiltonian of the electron is given by § 6 eq. (22)

$$H = e\phi + [\mu^2 + (\mathbf{p}-e\mathbf{A})^2]^{\frac{1}{2}} = E, \quad \mu = mc^2. \tag{12}$$

This can also be written in the form

$$E - e\phi = \frac{1}{c}(\mathbf{v}, \mathbf{p}-e\mathbf{A}) + \mu\sqrt{(1-v^2/c^2)}, \tag{13}$$

where

$$\frac{v_x}{c} = \frac{p_x - eA_x}{[\mu^2 + (\mathbf{p}-e\mathbf{A})^2]^{\frac{1}{2}}} \tag{14}$$

the electron, in order to get a finite result. This procedure is equivalent to the 'method of intermediate states' which we shall actually apply. In this method the emission of radiation and the deflexion of the electron are considered in two subsequent steps, the latter being due to the interaction V of the electron with the nucleus only. Plane waves can again be inserted in the matrix elements of V. We shall call this approximation '*Born's approximation*' also. (Compare subsection 3 and § 17.)

represents the velocity. In the quantum theory, according to Dirac,† (13) has to be written in the following form:

$$(E-e\phi)\psi = [(\boldsymbol{\alpha}, \mathbf{p}-e\mathbf{A})+\beta\mu]\psi, \tag{15}$$

where the vector $\boldsymbol{\alpha}$ and β are *matrices* which satisfy the commutation relations

$$\left.\begin{aligned} \alpha_x\beta+\beta\alpha_x = 0, \quad \alpha_x\alpha_y+\alpha_y\alpha_x = 0 \\ \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1. \end{aligned}\right\} \tag{16}$$

In (15) E and $\mathbf{p}$ must of course be replaced by the operators (2). Comparing (15) with (13) we see that the *matrix-vector* $\boldsymbol{\alpha}$ *represents the velocity* (divided by c) and β the quantity $\sqrt{(1-v^2/c^2)}$.

The vector $\boldsymbol{\alpha}$ and β can be represented as matrices with 4 rows and columns

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{17 a}$$

where $\boldsymbol{\sigma}, 1, -1, 0$ are matrices with two rows and columns

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{17 b}$$

$\boldsymbol{\sigma}$ is identical with Pauli's spin vector.

The wave function which satisfies Dirac's wave equation will contain, besides the variables t and $\mathbf{r}$, a variable s which can assume only four values corresponding to the four rows and columns of the operators (17).

For a free electron the wave equation

$$E\psi = [(\boldsymbol{\alpha}\mathbf{p})+\beta\mu]\psi \tag{18}$$

has the solution

$$\psi = u(s)e^{i(\mathbf{pr})/\hbar c - iEt/\hbar}, \quad E^2 = p^2+\mu^2, \tag{19}$$

where the amplitude $u(s)$ depends only on s and satisfies the same equation (18) p, E being now ordinary numbers. $u(s)$ has to be normalized so that

$$\sum_{s=1}^{4} |u(s)|^2 = 1. \tag{20}$$

For a given value of p and E the four linear and homogeneous equations for $u(s)$ (18) have *two independent solutions* $u_1(s)$ and $u_2(s)$ which correspond to the *two directions of spin* (in eq. (22) denoted by $\uparrow$ and $\downarrow$).

The energy E, however, is not determined uniquely by the momentum p as the square root

$$E = \pm\sqrt{(p^2+\mu^2)} \tag{21}$$

† P. A. M. Dirac, *Quantum Mechanics*, Oxford University Press, 2nd ed. 1935, Chap. XII.

may have two different signs. In classical dynamics only the positive energies have a physical significance. In quantum mechanics the *states of negative energy* cannot be excluded. They are connected, as we shall see in Chapter IV, with the existence of *positive electrons*. For a given momentum p therefore, we have altogether *four independent solutions* of (18), two with positive and two with negative energy. These four solutions are, for instance,

		$u(1)$	$u(2)$	$u(3)$	$u(4)$
$E>0$	$\uparrow$	1	0	$\frac{p_z}{\mu+E}$	$\frac{p_x-ip_y}{\mu+E}$
	$\downarrow$	0	1	$\frac{p_x+ip_y}{\mu+E}$	$-\frac{p_z}{\mu+E}$
$E<0$	$\uparrow$	$-\frac{p_z}{\mu+\lvert E\rvert}$	$-\frac{p_x-ip_y}{\mu+\lvert E\rvert}$	1	0
	$\downarrow$	$-\frac{p_x+ip_y}{\mu+\lvert E\rvert}$	$\frac{p_z}{\mu+\lvert E\rvert}$	0	1

$$\times\left[1+\frac{p^2}{(\mu+|E|)^2}\right]^{-\frac{1}{2}}. \tag{22}$$

In all applications of relativistic quantum mechanics to radiation problems we shall find it necessary to evaluate matrix elements of operators which are defined as products of the matrices α, β. If O is such an operator, the matrix elements are

$$(u^*, Ou') = \sum_{s=1}^{4} u^*(s) Ou'(s), \tag{23}$$

where u, u' represent the amplitudes of two states with the momenta $\mathbf{p}, \mathbf{p}'$ and certain spin directions and signs of energy. O operates on the four components of $u'(s)$. For the matrix element (23) we have the general rule

$$(u^*, Ou') = ((O^\dagger u)^*, u'), \tag{24}$$

where the operator $O^\dagger$ is derived from O by changing the order of factors (if, for instance, $O = \alpha_x \beta \alpha_y$, then $O^\dagger = \alpha_y \beta \alpha_x$).

Furthermore, we quote a few simple formulae which will be used frequently. If we denote by $\sum^p$ the summation over all four states having the same momentum p (summation over both spin directions and both signs of energy) we have, according to general rules of quantum mechanics,

$$\sum^{p'} (u^*, Ou')(u'^*, Qu'') = (u^*, OQu''). \tag{25}$$

The average value of O for a given state u is given by (u^*, Ou). The sum of the average values of O over all states with the same

momentum gives simply the 'spur' (diagonal sum) of the operator

$$\sum^p (u^*, Ou) = \mathrm{Sp}\, O. \tag{26}$$

$\mathrm{Sp}\, O$ can usually be evaluated without difficulty. From (17) we find, for instance, immediately

$$\mathrm{Sp}\,\alpha_x = \mathrm{Sp}\,\alpha_y = \mathrm{Sp}\,\alpha_z = \mathrm{Sp}\,\beta = 0 \tag{27 a}$$

$$\mathrm{Sp}\,\alpha_x\alpha_y = \mathrm{Sp}\,\alpha_x\beta = 0; \quad \mathrm{Sp}\,\alpha_x^2 = \mathrm{Sp}\,\beta^2 = 4 \tag{27 b}$$

$$\mathrm{Sp}\,\alpha_x\alpha_y\alpha_z = \mathrm{Sp}\,\alpha_x\alpha_y\beta = \ldots = 0. \tag{27 c}$$

In $\mathrm{Sp}\, O$ the order of the factors $\alpha_x, \beta \ldots$ of O can be changed cyclically. Making use of the commutation relations (16), the $\mathrm{Sp}\, O$ of an operator with a large number of factors may also be evaluated. One can then easily prove that if O contains an odd number of factors α_x, or of α_y, or of α_z, or of β, $\mathrm{Sp}\, O$ vanishes. The only operators containing four factors, for which $\mathrm{Sp}\, O \neq 0$, are

$$\begin{aligned} \mathrm{Sp}\,\alpha_x\alpha_y\alpha_x\alpha_y &= -\mathrm{Sp}\,\alpha_x^2\alpha_y^2 = -4 \\ \mathrm{Sp}\,\alpha_x\beta\alpha_x\beta &= -\mathrm{Sp}\,\alpha_x^2\beta^2 = -4. \end{aligned} \tag{27 d}$$

3. *Perturbation Theory.* The equations describing the behaviour of an electron interacting with a radiation field are far too complicated to be solved exactly. In all applications of the theory, therefore, the interaction energy is treated as small, and approximate solutions are obtained which are correct only to the first order in this energy. Apart, however, from the mathematical difficulties of proceeding to a higher degree of approximation it appears, as we shall see, that only the first order approximation has physical significance; the higher orders do not correspond to reality. This corresponds to a deep-seated limitation of the present theory.

The kind of perturbation theory applied is the same for all radiation processes, and will be developed here for the general case.

We assume that the Hamiltonian H of the system can be written in the form

$$H = H_0 + H', \tag{28}$$

where H' represents a perturbing term which is small compared with H_0. H' does not contain the time explicitly. If E_n, ψ_n represent the eigenvalues and eigenfunctions of the unperturbed system

$$E_n\psi_n = H_0\psi_n. \tag{29}$$

We can develop the solution of the actual Schrödinger equation (28)

$$-i\hbar\frac{\partial\psi}{\partial t} = (H_0 + H')\psi \tag{30}$$

in a series of the eigenfunctions of H_0:

$$\psi = \sum_n b_n(t)\psi_n e^{iE_nt/\hbar}. \tag{31}$$

The coefficients $b_n(t)$ are functions only of the time and the ψ_n of the coordinates of the unperturbed system (including the spin variables). $b_n(t)$ has the following physical significance: if we measure the energy of the unperturbed system† at the time t, the probability of finding the eigenvalue E_n is given by $|b_n(t)|^2$. Inserting (31) into (30), multiplying by ψ_n^* say, and integrating over the whole of the coordinate space,‡ we obtain a system of differential equations for the $b_n(t)$

$$-i\hbar\dot{b}_n(t) = \sum_m H'_{nm} b_m(t) e^{i(E_m-E_n)t/\hbar}, \tag{32}$$

where H'_{nm} represents the matrix element

$$H'_{nm} = \int \psi_n^* H' \psi_m. \tag{33}$$

The equations (32) are still exact, no terms having been neglected. We require a solution of (32) with specified initial conditions at the time $t = 0$. It will be convenient to choose these initial conditions so that, for $t = 0$, the system is found in a definite unperturbed state, E_{n_0} say. Then we have

$$b_n(0) = 0 \quad \text{except} \quad b_{n_0}(0) = 1. \tag{34}$$

Owing to the perturbation H'§ transitions to other states n take place, and after a time t the probability that a transition to a state n has occurred is equal to $|b_n(t)|^2$. If, however, H' is small, and if we choose the time t to be not too long, these transition probabilities will be small. We may then obtain an approximate solution of (32) by inserting in the right-hand side of (32) the values (34) of the $b_m(t)$ for $t = 0$. The solution which satisfies the initial conditions is given by

$$b_n(t) = \frac{H'_{nn_0}(e^{i(E_{n_0}-E_n)t/\hbar}-1)}{E_{n_0}-E_n}. \tag{35}$$

The probability of finding the state E_n at the time t is equal to

$$|b_n(t)|^2 = \frac{2|H'_{nn_0}|^2}{(E_{n_0}-E_n)^2}\left(1-\cos(E_{n_0}-E_n)\frac{t}{\hbar}\right). \tag{36}$$

It is proportional to the square of the matrix element H'_{nn_0}.

† For this purpose it is necessary to switch off the perturbation H' at the time t. This 'switching off' is a necessary part of the measurement.

‡ This integration includes the summation over the spin variables.

§ Which has to be switched on at $t = 0$, see footnote † above.

In many cases, however, the matrix elements H'_{nn_0} for the transition from an initial state n_0 to a final state n *vanish*. If this is the case the method of approximation has to be refined. We shall assume that there exist some '*intermediate states* n'' for which $H'_{nn'}$ as well as $H'_{n'n_0}$ are different from zero. In the first approximation, on the right-hand side of (32), we may equate to zero all $b_m(t)$ except $b_{n'}$ and b_{n_0}. b_{n_0} may again be equated to 1. The equations

$$-i\hbar\dot{b}_{n'} = H'_{n'n_0}\, e^{i(E_{n_0}-E_{n'})t/\hbar} \tag{37 a}$$

$$-i\hbar\dot{b}_{n} = \sum_{n'} H'_{nn'} b_{n'}(t) e^{i(E_{n'}-E_{n})t/\hbar} \tag{37 b}$$

have the solution (with the correct initial condition)

$$b_n(t) = \sum_{n'} \frac{H'_{nn'}H'_{n'n_0}}{E_{n_0}-E_{n'}}\left[\frac{e^{i(E_{n_0}-E_n)t/\hbar}-1}{E_{n_0}-E_n} - \frac{e^{i(E_{n'}-E_n)t/\hbar}-1}{E_{n'}-E_n}\right]. \tag{38}$$

The transition probability is given by

$$|b_n(t)|^2 = \frac{2|H'|^2}{(E_{n_0}-E_n)^2}\left(1-\cos(E_{n_0}-E_n)\frac{t}{\hbar}\right)+$$

$$+\text{terms containing } (E_{n'}-E_n), \tag{39 a}$$

with

$$|H'| = \sum_{n'} \frac{H'_{nn'}H'_{n'n_0}}{E_{n_0}-E_{n'}}. \tag{39 b}$$

The matrix-element $|H'|$ is here quadratic in the perturbation energy H'. The terms 'containing $E_{n'}-E_n$' play no role in the case to be considered now.

In all radiation problems either the initial or the final state belongs to the continuous spectrum i.e. a very large number of states exist which have the same or nearly the same energy, momentum, etc. (light quantum in a volume L^3). We assume therefore that in the neighbourhood of the final state n there are a large number of states with the same physical properties and that $\rho_E dE$ represents the number of these states with energy between E and $E+dE$. H' will in general only depend upon the physical properties of the final state (for instance on the energy E) but not upon n. Then we are not interested in the probability of finding our system, after the time t, only in a particular state E_n, but rather in *any one* of these states. We obtain this probability by multiplying (36), (39) by $\rho_E dE$ and integrating over a small energy interval ΔE.†

† Compare the very similar case treated in § 6 subsection 5, emission of light.

These integrals have the form (writing E_0 for E_{n_0} and E for E_n)

$$\int_{\Delta E} \frac{f(E)(1-\cos(E_0-E)t/\hbar)}{(E_0-E)^2}\,dE. \tag{39}$$

If $t \gg \hbar/E_0$, the integrand has a strong maximum for $E_0 = E$. The probability is appreciable only when the *energy of the final state is equal to the energy of the initial state*, i.e. when

$$E_0 = E. \tag{40}$$

(40) expresses the law of *conservation of energy*. The *energy* is therefore *conserved for all transitions from or into the continuous spectrum*. For the transitions from or to the intermediate states n' the energy is, of course, in general not conserved.

$f(E)$ will be in general a slowly varying function. Then the integral (39) becomes

$$\int_{\Delta E} \frac{f(E)(1-\cos(E_0-E)t/\hbar)}{(E_0-E)^2}\,dE = f(E_0)\frac{\pi}{\hbar}t. \tag{41}$$

The terms 'containing $E_{n'}-E_n$' of (39) give no contribution to this integral if $E_n \neq E_{n'}$. The *case of resonance* $E_n = E_{n'}$ has to be considered separately (see for instance § 15).

Thus, if t is not too long (but $t \gg \hbar/E_0$), the transition probability is proportional to t. We can therefore define a *transition probability per unit time* $w_{E_0 E}$ which according to (36), (39), (41) is given by

$$w_{E_0 E} = \frac{1}{t}\int |b_n(t)|^2 \rho_E\,dE = \frac{2\pi}{\hbar}\rho_{E_0}|H'(E_0)|^2. \tag{42}$$

For the matrix element $H'(E_0)$ one has to insert

$$H'(E_0) = H'_{n_0 n} \tag{43 a}$$

if a direct transition n_0 to n is possible, or

$$H'(E_0) = \sum_{n'} \frac{H'_{n_0 n'} H'_{n' n}}{E_{n_0}-E_{n'}} \tag{43 b}$$

if it occurs only by passing through the intermediate states n'. The generalization of (43) when two successive intermediate states are necessary obviously gives

$$H'(E_0) = \sum_{n' n''} \frac{H'_{n_0 n'} H'_{n' n''} H'_{n'' n}}{(E_{n_0}-E_{n'})(E_{n_0}-E_{n''})}. \tag{43 c}$$

(42) holds also when the initial state belongs to the continuous spectrum, $\rho_{E_0}\,dE_0$ being then the number of initial states with an energy between E_0 and E_0+dE_0.

As it is seen immediately from the expressions (43 a)–(43 c) for the matrix element $|H'|$, the perturbation theory developed in this section means an expansion of the transition probability in powers of the perturbation function H'.

10. General theory of the interaction of light with particles

1. *The interaction function.* We shall now apply the general quantum theory of the field to the problem of the interaction of light with charged particles. The Hamiltonian describing the motion of the particles and the field is given by § 7 eq. (26). Writing that part $\sum H_k$ which depends upon the coordinates of the particles in linear form introduced by Dirac, § 9 eq. (15), the total Hamiltonian becomes

$$H = \sum_k [(\boldsymbol{\alpha}_k, \mathbf{p}_k - e\mathbf{A}) + \beta_k \mu_k + e_k \phi^e] + \frac{1}{2} \sum_{i \neq k} \frac{e_i e_k}{r_{ik}} + \sum_\lambda n_\lambda k_\lambda, \qquad (1)$$

where A represents the vector potential of the external field and the radiation (not including the longitudinal part of the field) and ϕ^e the scalar potential of the external field only. $\boldsymbol{\alpha}_k$, β_k are the Dirac matrices § 9 eq. (17) for the kth particle. The last term represents the energy of the radiation.

In equation (1) we can separate those terms which depend on the coordinates of the particles as well as those of the field, since (1) is now a linear function of the field. These terms will represent the *interaction of the field with the particles.* Denoting the vector potential of the radiation field simply by A and the vector potential of the external field by $\mathbf{A}^e$ we can write instead of (1):

$$H = H_0 + H' \qquad (2\,a)$$

$$H_0 = \sum_k [(\boldsymbol{\alpha}_k, \mathbf{p}_k - e_k \mathbf{A}^e) + \beta_k \mu_k + e_k \phi^e] + \frac{1}{2} \sum_{i \neq k} \frac{e_i e_k}{r_{ik}} + \sum_\lambda n_\lambda k_\lambda$$
$$= H_p + H_r \qquad (2\,b)$$

$$H' = -\sum_k e_k (\boldsymbol{\alpha}_k \mathbf{A}(k)). \qquad (2\,c)$$

H_0 consists of two parts: (i) the energy H_p of the particles including their static interaction, (ii) the energy of the radiation field

$$H_r = \sum n_\lambda k_\lambda.$$

H' depends on the coordinates of the particles as well as the variables describing the field, and represents therefore the desired interaction function.

Using for H_k the non-relativistic approximation

$$H_k = \frac{1}{2\mu_k}(\mathbf{p}_k - e_k\mathbf{A})^2,$$

H' becomes $$H' = -\sum_k \left[\frac{e_k}{\mu_k}(\mathbf{p}_k\mathbf{A}(k)) - \frac{e_k^2}{2\mu_k}A^2(k)\right]. \qquad \text{N.R.} \quad (3)$$

According to § 7 eq. (3) $\mathbf{A}$ can be represented as a series of plane waves:

$$\mathbf{A} = \sum_\lambda (q_\lambda \mathbf{A}_\lambda + q_\lambda^* \mathbf{A}_\lambda^*) \qquad (4\,\text{a})$$

$$\mathbf{A}_\lambda = \mathbf{e}_\lambda \sqrt{(4\pi c^2)} e^{i(\kappa_\lambda r)} \quad (\kappa_\lambda = \nu_\lambda/c), \qquad (4\,\text{b})$$

where the q_λ are represented as matrices (§ 7 eq. (9))

$$e^{-i\nu_\lambda t} q^*_{n_\lambda+1, n_\lambda} = e^{i\nu_\lambda t} q_{n_\lambda, n_\lambda+1} = \sqrt{\left(\frac{\hbar(n_\lambda+1)}{2\nu_\lambda}\right)} \qquad (4\,\text{c})$$

(all other matrix elements vanish). $\mathbf{e}_\lambda$ is the unit vector in the direction of polarization. q_λ^* has matrix elements only for transitions in which n_λ decreases by one, i.e. in which *one light quantum k_λ is absorbed.* The matrix elements of q_λ correspond to an emission of one quantum k_λ.

In all applications of the theory to radiation processes we shall consider the *interaction H'* as a *small perturbation.* We can then apply the general perturbation theory § 9 subsection 3. From the point of view of this perturbation theory, the interaction H' causes *transitions of the unperturbed system* (particles+radiation), which in general will be connected with a change of the quantum numbers n_λ, i.e. with emission and absorption of light quanta. The condition for the existence of a *transition probability per unit time*, in this case, is always satisfied, since the radiation field has a continuous spectrum. From this it follows also that the *energy* of the unperturbed system is *conserved for all* these *radiation processes.*

We shall calculate the transition probabilities as far as the first non-vanishing order in H'. The same approximation has in fact already been made in the classical theory. This can clearly be seen for instance from § 6 subsection 5 where we have calculated the emission of light by an oscillator by a method which corresponds exactly to the quantum theoretical method.

The higher terms of the expansion in H' would involve the higher terms of the *radiative reaction forces.* In the classical theory (§ 4) it has been shown that the latter are in general *small* compared with

the other forces to which the electron is subjected. The exact condition for this has been formulated for an oscillator in § 4 eq. (7'):

$$\lambda \gg r_0 = e^2/mc^2. \tag{5}$$

(5) can also be expressed in the following way: The *radiative lifetime*† *must be large compared with the period* $1/\nu$ *of the oscillator* (*or of the light wave emitted*).

In the quantum theory this condition is necessary for the *existence of stationary states*, since no stationary states would exist if the transition probability were of the same order of magnitude as the period of revolution of the electron. We have already made use of this condition in the computation of the transition probabilities § 9.3 ($t \gg \hbar/E_0$).

In the classical theory we have given for the reaction force an approximate expression which can be considered as an expansion in powers of the ratio $\bar{r}_0/\lambda$ (§ 4, subsection 2). The higher terms of this expansion depend upon the *structure of the electron* and have therefore no physical meaning. They are small and decrease rapidly with $\bar{r}_0/\lambda$. In the quantum theory, however, the situation is more serious. As we shall see later (§ 18), the higher approximations in H', which would give a detailed description of the radiative reaction, *diverge*. Therefore it is of decisive importance to emphasize that from the correspondence to the classical theory we may not expect these higher approximations to have any physical meaning, since the analogous terms in the classical theory would depend upon the structure of the electron. The only part of the reaction force which is independent of the radius $\bar{r}_0$ of the electron is the term proportional to $\ddot{\mathfrak{v}}$, which is responsible for the natural line breadth. By treating radiative processes only up to the first approximation we neglect the damping entirely. We shall see, however (§§ 12, 15), that the damping can also be treated in quantum theory by an improvement of the mathematical solution, going beyond the first approximation to the same extent as it corresponds to the classical $\ddot{\mathfrak{v}}$-term. In § 25 we shall give a general formulation of the theory which includes the damping to the said extent, and is yet free of the above divergence difficulties.

In general the damping is small and can be neglected except for questions concerning the line breadth. The first approximation then always suffices. In quantum theory not even the condition (5) is

† Inverse of the radiative transition probability.

necessary for this purpose and we shall see in § 25 that the damping remains even small if $\lambda < e^2/mc^2$. This is so because quantum theory becomes effective *before* the damping, namely, at wave-lengths much larger than r_0, viz.

$$\lambda \sim \hbar/mc = 137 r_0. \tag{6}$$

2. *Matrix elements.* To compute the transition probabilities of radiation processes according to the general perturbation theory § 9 subsection 3, we must work out the matrix elements of the interaction H'. According to (2 b) the Hamiltonian of the unperturbed system H_0 consists of two parts H_p and H_r, where H_p depends only upon the coordinates of the particles. The radiation field is described by the number n_λ of light quanta for each radiation oscillator. If E_a, ψ_a represent a solution of the Schrödinger equation for the particles (including their static interaction)

$$E_a \psi_a = H_p \psi_a,$$

a state of the unperturbed system will be determined by (i) the quantum state a of the particles, and (ii) by the number of light quanta n_λ. The total energy of this state is

$$E = E_a + \sum_\lambda n_\lambda k_\lambda. \tag{7}$$

According to the general perturbation theory § 9, the interaction H' between the particles and the field will cause transitions of the unperturbed system. Such a transition will in general consist of a transition $a \to b$ of the state of the particles and of a change of the number of light quanta n_λ, i.e. of an *emission and absorption of light quanta*

$$a, n_1, \ldots, n_\lambda, \ldots \to b, n_1', \ldots, n_\lambda', \ldots . \tag{8}$$

According to (2 c), (3), and (4 c), the matrix elements of H' for the transition (8) can easily be obtained. Since the matrix elements of q_λ, q_λ^* are different from zero only if n_λ changes by one unit, and since H' is a linear function of the q_λ, the matrix elements of H' are all zero, unless for one single oscillator, the λth say, $n_\lambda' = n_\lambda + 1$ or $n_\lambda - 1$. In the first case ($n_\lambda' = n_\lambda + 1$) the transition (8) corresponds to an *emission of a light quantum* k_λ, in the second case ($n_\lambda' = n_\lambda - 1$) to an *absorption*. For the emission, only the term $q_\lambda \mathbf{A}_\lambda$ of (4 a) gives a contribution to the matrix element (for the absorption only $q_\lambda^* \mathbf{A}_\lambda^*$).

According to (2 c) and (4 c) we obtain for the matrix elements of H' (omitting the time factors)

$$H'_{an_\lambda;bn_\lambda+1} = -\sqrt{\left(\frac{\hbar^2}{2k_\lambda}\right)}\sqrt{(n_\lambda+1)}\sum_k e_k \int \psi_a^*(\boldsymbol{\alpha}_k \mathbf{A}_\lambda(k))\psi_b \qquad (9\,a)$$

$$H'_{an_\lambda+1;bn_\lambda} = -\sqrt{\left(\frac{\hbar^2}{2k_\lambda}\right)}\sqrt{(n_\lambda+1)}\sum_k e_k \int \psi_a^*(\boldsymbol{\alpha}_k \mathbf{A}_\lambda^*(k))\psi_b, \qquad (9\,b)$$

where $\int$ denotes the integration over the space coordinates and summation over the spin variables s_k of all particles. Inserting for $\mathbf{A}_\lambda$, $\mathbf{A}_\lambda^*$ equation (4 b) we have for a single particle

$$H'_{an_\lambda;bn_\lambda+1} = -e\sqrt{\left(\frac{2\pi\hbar^2c^2}{k_\lambda}\right)}\sqrt{(n_\lambda+1)}\int \psi_a^*\,\alpha_e\, e^{i(\boldsymbol{\kappa}_\lambda \mathbf{r})}\psi_b \qquad (10\,a)$$

$$H'_{an_\lambda+1;bn_\lambda} = -e\sqrt{\left(\frac{2\pi\hbar^2c^2}{k_\lambda}\right)}\sqrt{(n_\lambda+1)}\int \psi_a^*\,\alpha_e\, e^{-i(\boldsymbol{\kappa}_\lambda \mathbf{r})}\psi_b, \qquad (10\,b)$$

where α_e represents the component of the matrix vector $\boldsymbol{\alpha}$ in the direction of polarization of the light quantum.

For the transitions (8) the energy will not in general be conserved if either of the states in question is an intermediate state. The energy differences of the two states for the two transitions are

$$E - E' = E_a - E_b \mp k_\lambda. \qquad (11)$$

In the non-relativistic approximation, the first term of (3) gives also matrix elements for transitions by which one quantum is emitted or absorbed. The second term is proportional to

$$A^2 = \sum_{\lambda,\mu}[q_\lambda q_\mu(\mathbf{A}_\lambda \mathbf{A}_\mu) + q_\lambda q_\mu^*(\mathbf{A}_\lambda \mathbf{A}_\mu^*) + q_\lambda^* q_\mu(\mathbf{A}_\lambda^* \mathbf{A}_\mu) + q_\lambda^* q_\mu^*(\mathbf{A}_\lambda^* \mathbf{A}_\mu^*)].$$

The matrix elements are different from zero if two quanta are emitted or absorbed (or one emitted, one absorbed). Thus (for one particle)

$$H'_{an_\lambda;bn_\lambda+1} = -\frac{e}{\mu}\sqrt{\left(\frac{2\pi\hbar^2c^2}{k_\lambda}\right)}\sqrt{(n_\lambda+1)}\int \psi_a^*\, p_e\, e^{i(\boldsymbol{\kappa}_\lambda \mathbf{r})}\psi_b \qquad \text{N.R.} \quad (12\,a)$$

$$H'_{an_\lambda+1;bn_\lambda} = -\frac{e}{\mu}\sqrt{\left(\frac{2\pi\hbar^2c^2}{k_\lambda}\right)}\sqrt{(n_\lambda+1)}\int \psi_a^*\, p_e\, e^{-i(\boldsymbol{\kappa}_\lambda \mathbf{r})}\psi_b, \quad \text{N.R.} \quad (12\,b)$$

where p_e represents the component of $\mathbf{p}$ in the direction of polarization, and

$$H'_{a,n_\lambda+1,n_\mu+1;b,n_\lambda,n_\mu}$$

$$= \frac{e^2}{\mu}(\mathbf{e}_\lambda \mathbf{e}_\mu)\frac{2\pi\hbar^2c^2}{\sqrt{(k_\lambda k_\mu)}}\sqrt{\{(n_\lambda+1)(n_\mu+1)\}}\int \psi_a^*\, e^{-i(\boldsymbol{\kappa}_\lambda+\boldsymbol{\kappa}_\mu, \mathbf{r})}\psi_b$$

$$H'_{a,n_\lambda+1,n_\mu;b,n_\lambda,n_\mu+1}$$

$$= \frac{e^2}{\mu}(\mathbf{e}_\lambda \mathbf{e}_\mu)\frac{2\pi\hbar^2c^2}{\sqrt{(k_\lambda k_\mu)}}\sqrt{\{(n_\lambda+1)(n_\mu+1)\}}\int \psi_a^* e^{i(\boldsymbol{\kappa}_\mu-\boldsymbol{\kappa}_\lambda,\mathbf{r})}\psi_b$$

$$H'_{a,n_\lambda,n_\mu;b,n_\lambda+1,n_\mu+1}$$

$$= \frac{e^2}{\mu}(\mathbf{e}_\lambda \mathbf{e}_\mu)\frac{2\pi\hbar^2c^2}{\sqrt{(k_\lambda k_\mu)}}\sqrt{\{(n_\lambda+1)(n_\mu+1)\}}\int \psi_a^* e^{i(\boldsymbol{\kappa}_\mu+\boldsymbol{\kappa}_\lambda,\mathbf{r})}\psi_b. \quad \text{N.R.} \quad (13)$$

3. *Conservation of momentum.* Finally, we consider the special but very important case of *free electrons*. The relativistic wave function for a free electron has been given in § 9 eqq. (3), (19). For two states ψ_a, ψ_b with the momenta $\mathbf{p}_a$, $\mathbf{p}_b$ we have

$$\psi_a = u_a(s)e^{i(\mathbf{p}_a\mathbf{r})/\hbar c}, \qquad \psi_b = u_b(s)e^{i(\mathbf{p}_b\mathbf{r})/\hbar c}.$$

Inserting these functions in the matrix elements (10) we obtain for the integral (putting $\boldsymbol{\varkappa} = \mathbf{k}/\hbar c$)

$$\int = (u_a^*, \alpha_e u_b)\int e^{i(\mathbf{p}_b-\mathbf{p}_a\pm\mathbf{k}_\lambda,\mathbf{r})/\hbar c}. \quad (14)$$

This integral obviously vanishes unless

$$\mathbf{p}_b-\mathbf{p}_a\pm\mathbf{k}_\lambda = 0, \quad (15)$$

for emission and absorption respectively.

Equation (15) expresses the law of *conservation of momentum*. For all interaction processes of a light quantum with a free particle, the momentum is therefore conserved, if we attribute to the light quantum a momentum $\mathbf{k}_\lambda$.† This fact completes the enumeration of the particle properties of a light quantum as quoted in § 7. On the other hand, we must remember that the conservation of energy was a general result of the perturbation theory, § 9 p. 90, which is valid for all transitions from or to the continuous spectrum and therefore for all radiation processes.

The matrix elements for transitions of a free electron for which the momentum is conserved are given, according to (10) and (14), by

$$H'_{p_a n_\lambda+1;\, p_b n_\lambda} = H'^*_{p_b n_\lambda;\, p_a n_\lambda+1} = -e\sqrt{\left(\frac{2\pi\hbar^2c^2}{k_\lambda}\right)}\sqrt{(n_\lambda+1)}(u_a^*\alpha_e u_b), \quad (16)$$

the wave functions of the plane waves being normalized for a unit volume.

† For a bound electron the momentum is not in general conserved since the nucleus can take any amount of momentum.

For the non-relativistic case the matrix elements are given according to (12) and (13) by

$$H'_{p_a n_\lambda+1;p_b n_\lambda} = H'^{*}_{p_a n_\lambda;p_b n_\lambda+1} = -e\sqrt{\left(\frac{2\pi\hbar^2c^2}{k_\lambda}\right)}\sqrt{(n_\lambda+1)}\frac{p_e}{\mu}, \quad \text{N.R. (17)}$$

and for transitions in which two quanta are involved, for instance, by

$$H'_{p_a,n_\lambda+1,n_\mu;p_b n_\lambda,n_\mu+1} = \frac{e^2}{\mu}(\mathbf{e}_\lambda \mathbf{e}_\mu)\frac{2\pi\hbar^2c^2}{\sqrt{(k_\lambda k_\mu)}}\sqrt{\{(n_\lambda+1)(n_\mu+1)\}}. \quad \text{N.R. (18)}$$

4. *Classification of radiation processes.* As has been explained in subsection 1, our theory of the radiation processes will be based on an expansion in powers of H'. Since the (relativistic) matrix elements of H' are proportional to e we can consider this expansion alternatively as an expansion in powers of e. On the other hand, the matrix elements of H' are different from zero only for those transitions for which only one light quantum is emitted or absorbed. A process in which two light quanta are involved (as for instance the scattering where one quantum is absorbed and another one emitted) can therefore only occur by passing through an intermediate state. The latter has to be chosen so that, in the transition from the initial state to the intermediate state, only one quantum is absorbed (or emitted), whereas the second quantum is emitted (or absorbed) during the second step, intermediate state → final state. The transition probabilities for these processes are proportional to H'^4 (§ 9 eqq. (42), (43 b)), i.e. to e^4. We can therefore classify all radiation processes according to the power of e^2 or to the number of light quanta involved. The transition probability for a process in which n quanta are emitted or absorbed will be proportional to e^{2n}.†

In some cases it is convenient to consider also the Coulomb interaction between two free particles as a perturbation causing a deflexion of the particles (for instance for the 'Bremsstrahlung'). The matrix elements of the Coulomb interaction are proportional to e^2 (cf. eq. (22)). In the power of e involved the latter is equivalent to an emission or absorption of two light quanta. If we now classify the radiation processes according to the power of e, we obtain the following scheme:

A. First order. $w \sim e^2$. One light quantum: emission, absorption, one quantum annihilation of positrons.

† This classification, of course, remains unchanged if we use the non-relativistic interaction (3). The second term has indeed matrix elements for transitions in which two quanta are involved. These matrix elements are just proportional to e^2.

B. Second order. $w \sim e^4$. Two light quanta: dispersion, Raman effect, two quanta annihilation of positrons.

C. Third order. $w \sim e^6$. Three light quanta or one light quantum and Coulomb deflexion:† 'Bremsstrahlung', creation of pairs by a light quantum in the field of a nucleus. Double scattering (see § 18).

5. *Retarded interaction of two particles.*‡ Before discussing these radiation processes we shall consider briefly the interaction of two particles in the first approximation of an expansion in powers of e (or Ze). The first non-vanishing term in this expansion represents the approximation which Born used for the treatment of collision problems (compare footnote ‡ page 83). As we have pointed out in § 6 and § 7 the interaction of two particles occurs in two different terms in the total Hamiltonian of the system (1):

The *static interaction* of two particles with charges $Z_1 e$ and $Z_2 e$,

$$V = \frac{Z_1 Z_2 e^2}{r_{ik}}, \tag{19}$$

is contained as a separate term in the Hamiltonian. The effect of *retardation*, on the other hand, appears as a simultaneous emission and absorption of light quanta by the two particles and is therefore contained in the term describing the interaction of the particles with *light*.

We shall confine ourselves to the important case of two free particles. Their relativistic wave-functions are given by § 9 eq. (19).

The interaction V can now cause transitions from an initial state A in which the particles have momenta $\mathbf{p}_{01}$, $\mathbf{p}_{02}$ to a final state F with the momenta $\mathbf{p}_1$, $\mathbf{p}_2$. In the first approximation of an expansion in powers of e the transition probability is given by the

† For the collision of an electron with a nucleus of charge Z, the Coulomb interaction is proportional to e^2Z. The expansion mentioned above is then actually an expansion in terms of e^2Z. The first term represents Born's approximation (§ 9 (9)). For some problems it is necessary to distinguish between the expansions in e and in Ze^2, because it may happen that the first approximation in e^2Z is very inaccurate, whereas for the interaction with light the first power of e may still be used (see for instance § 17, 'Bremsstrahlung').

‡ The retarded interaction of two particles has been studied by: G. Breit, *Phys. Rev.* **34** (1929), 553; ibid. **39** (1932), 616; C. Møller, *Zs. f. Phys.* **70** (1931), 686; *Ann. d. Phys.* **14** (1932), 531; L. Rosenfeld, *Zs. f. Phys.* **73** (1931), 253; H. Bethe and E. Fermi, ibid. **77** (1932), 296; W. Heitler and L. Nordheim, *Journ. d. Phys.* **5** (1934), 449; H. R. Hulme, *Proc. Roy. Soc.* **154** (1936), 487.

The treatment of the problem by an expansion of the interaction in powers of e (which we follow in this book) is due to Møller.

matrix element of the interaction V:

$$V_{AF} = Z_1 Z_2 e^2 \int \frac{1}{r_{12}} e^{i[(\mathbf{p}_{01}-\mathbf{p}_1, \mathbf{r}_1)+(\mathbf{p}_{02}-\mathbf{p}_2, \mathbf{r}_2)]/\hbar c} (u_{01}^* u_1)(u_{02}^* u_2), \tag{20}$$

where u_{01}, u_1,... represent the Dirac amplitudes for the states with momenta $\mathbf{p}_{01}, \mathbf{p}_1$.

V_{AF} vanishes unless the momentum is conserved

$$\mathbf{p}_{01}+\mathbf{p}_{02}-\mathbf{p}_1-\mathbf{p}_2 = 0. \tag{21}$$

In this case the integral (20) has the value

$$V_{AF} = \frac{4\pi\hbar^2 c^2 Z_1 Z_2 e^2}{|\mathbf{p}_{01}-\mathbf{p}_1|^2} (u_{01}^* u_1)(u_{02}^* u_2). \tag{22}$$

For the present we do not assume that the energy is conserved also in the transition $A \to F$, because one of the states A or F may be supposed to represent only an intermediate state in a more complicated process.

V_{AF} is proportional to $Z_1 Z_2 e^2$. As mentioned in subsection 4, the Coulomb interaction represents, with respect to the power of e, a perturbation of *second order*. From this point of view the transition $\mathbf{p}_{01}, \mathbf{p}_{02} \to \mathbf{p}_1, \mathbf{p}_2$ caused by the static interaction is equivalent to a transition in which two light quanta are emitted or absorbed.

In fact, the transition $A \to F$ in question can also be caused, to the same order of approximation, by the interaction of the particles with *light*.

Since the momentum also is then conserved, we can introduce an intermediate state I in which the first particle has made the transition $\mathbf{p}_{01} \to \mathbf{p}_1$ by emitting a light quantum $\mathbf{k}$:

$$\mathbf{p}_{01}-\mathbf{p}_1 = \mathbf{k}. \tag{23}$$

In the transition to the final state F this light quantum is absorbed by the second particle

$$-\mathbf{k} = \mathbf{p}_{02}-\mathbf{p}_2 = -(\mathbf{p}_{01}-\mathbf{p}_1). \tag{24}$$

The matrix element for this transition (with respect to the power of e, Z_1, Z_2) is of the same order as that of the direct Coulomb interaction. It is given (according to § 9 (43 b)) by (dropping the dashes of H')

$$H_{AF} = \frac{H_{AI} H_{IF}}{E_A - E_I}, \tag{25}$$

where E_A, E_I represent the energies of the states and

$$\left.\begin{aligned} H_{AI} &= -eZ_1\sqrt{\left(\frac{2\pi\hbar^2c^2}{k}\right)}(u_{01}^*\,\alpha_1\,u_1), \\ H_{IF} &= -eZ_2\sqrt{\left(\frac{2\pi\hbar^2c^2}{k}\right)}(u_{02}^*\,\alpha_2\,u_2). \end{aligned}\right\} \tag{26}$$

α_1, α_2 represent the components, in the direction of polarization of $\mathbf{k}$, of the matrix vectors $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ for the two particles.†

The transition can take place by passing through a second intermediate state II differing from I by an interchange of the roles of the two particles. A light quantum $\mathbf{k}'$ is emitted first by the second particle

$$\mathbf{k}' = \mathbf{p}_{02} - \mathbf{p}_2 = -\mathbf{k} \tag{27}$$

which is then absorbed by the first particle. The matrix elements are

$$\left.\begin{aligned} H_{AII} &= -eZ_2\sqrt{\left(\frac{2\pi\hbar^2c^2}{k}\right)}(u_{02}^*\,\alpha_2\,u_2), \\ H_{IIF} &= -eZ_1\sqrt{\left(\frac{2\pi\hbar^2c^2}{k}\right)}(u_{01}^*\,\alpha_1\,u_1). \end{aligned}\right\} \tag{28}$$

Since $\mathbf{k}' = -\mathbf{k}$ the directions of polarization for $\mathbf{k}'$ are the same as for $\mathbf{k}$.

The total matrix element for the transition $A \to F$ consists now of two terms. The first term, due to the Coulomb interaction, is given by (22). The second term due to the interaction with light is given according to § 9 eq. (43 b) by

$$\sum \frac{H_{AI}\,H_{IF}}{E_A - E_I}, \tag{29}$$

where the sum $\sum$ has to be taken over all intermediate states, i.e. in our case over the states I and II, and, of course, over all directions of polarization of the quantum $\mathbf{k}$, because for each direction of polarization we have another intermediate state. The energy differences $E_A - E_I$, etc., are, according to (23) and (27),

$$\left.\begin{aligned} E_A - E_I &= \epsilon_{01} - \epsilon_1 - k, \\ E_A - E_{II} &= \epsilon_{02} - \epsilon_2 - k, \end{aligned}\right\} \tag{30}$$

† In (26) it is assumed that both particles satisfy Dirac's relativistic wave equation. As far as is known at present this is only true for electrons and positive electrons. For protons it is not quite certain that Dirac's equation holds. Then the non-relativistic matrix elements must be used instead of (26). The effect of retardation can then be taken into account only up to terms in v/c.

where ϵ_{01}, ϵ_1 denote the energies of the first particle in the state with the momenta p_{01}, p_1. Hence the complete matrix element becomes

$$H_{AF} = V + \sum \frac{H_{AI} H_{IF}}{E_A - E_I} = \frac{4\pi\hbar^2 c^2 Z_1 Z_2 e^2}{k^2} \times$$

$$\times \left[(u_{01}^* u_1)(u_{02}^* u_2) + \frac{k}{2} S(u_{01}^* \alpha_1 u_1)(u_{02}^* \alpha_2 u_2) \left(\frac{1}{\epsilon_{01} - \epsilon_1 - k} + \frac{1}{\epsilon_{02} - \epsilon_2 - k} \right) \right], \tag{31}$$

where S denotes the summation over both directions of polarization of $\mathbf{k}$.†

(31) represents the general matrix element for the retarded interaction of two particles. The effect of retardation is obviously contained in the second term.

We consider now the special case where the energy is also conserved. We have then an *elastic collision* between two free particles. The conservation of energy states that

$$\epsilon_{01} - \epsilon_1 = -(\epsilon_{02} - \epsilon_2) = \epsilon. \tag{32}$$

The summation S over the directions of polarization can easily be carried out. If we choose for the z-direction the direction of $\mathbf{k}$ we have for two vectors $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$

$$S\alpha_1 \alpha_2 = \alpha_{1x} \alpha_{2x} + \alpha_{1y} \alpha_{2y} = (\boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2) - \frac{(\boldsymbol{\alpha}_1 \mathbf{k})(\boldsymbol{\alpha}_2 \mathbf{k})}{k^2} \tag{33}$$

Making use of Dirac's wave equation we can write

$$\{u_{01}^* (\boldsymbol{\alpha}_1 \mathbf{k}) u_1\} = \{u_{01}^* (\boldsymbol{\alpha}_1, \mathbf{p}_{01} - \mathbf{p}_1) u_1\} = (\epsilon_{01} - \epsilon_1)(u_{01}^* u_1) = \epsilon(u_{01}^* u_1),$$
$$\{u_{02}^* (\boldsymbol{\alpha}_2 \mathbf{k}) u_2\} = \epsilon(u_{02}^* u_2), \tag{34}$$

and obtain for (31) simply

$$H = \frac{4\pi\hbar^2 c^2 e^2 Z_1 Z_2}{k^2 - \epsilon^2} [(u_{01}^* u_1)(u_{02}^* u_2) - (u_{01}^* \boldsymbol{\alpha}_1 u_1)(u_{02}^* \boldsymbol{\alpha}_2 u_2)]. \tag{35}$$

(The second term has to be understood as a scalar product:

$$(u_{01}^* \alpha_{1x} u_1)(u_{02}^* \alpha_{2x} u_2) + (u_{01}^* \alpha_{1y} u_1)(u_{02}^* \alpha_{2y} u_2) + \dots .)$$

The formula (35), which was first deduced by Møller (loc. cit.), differs from the formula (22) for the non-retarded interaction in the following two points: (i) there is an additional term $(u_{01}^* \boldsymbol{\alpha}_1 u_1)(u_{02}^* \boldsymbol{\alpha}_2 u_2)$ in the interaction—this term gives simply the spin interaction between the two particles; (ii) the denominator $|\mathbf{p}_{01} - \mathbf{p}_1|^2$ of (22) is replaced

† If the two particles are equal the effect of 'Austausch' has been neglected in the matrix element (31).

by $k^2-\epsilon^2 = |\mathbf{p}_{01}-\mathbf{p}_1|^2-(\epsilon_{01}-\epsilon_1)^2$, i.e. by the square of the *relativistically invariant 4-vector* $(p_{0i}-p_i)^2$ of *energy and momentum* (see § **2** subsection 3). Since the numerator is also invariant (as can easily be shown from Dirac's equation) the whole matrix element (35) is invariant.

The expression (35) can also be obtained by computing directly in a suitable way the matrix element of the retarded interaction of the two particles as given by the retarded potentials (§ **1** eq. (13)) which corresponds to the transition $\mathbf{p}_{01} \to \mathbf{p}_1$ and $\mathbf{p}_{02} \to \mathbf{p}_2$. In this way (35) was deduced by Møller (loc. cit.).

In the present theory the retarded interaction between two electrons has only been taken into account in the first approximation ($\sim e^4$). An exact theory of the effect of retardation is not however possible. We shall see in § **18** that it is a general feature of the theory that the interaction of light with an electron diverges in all higher approximations. Therefore, the effect of retardation between two electrons can also only be considered up to terms proportional to e^4. It is, however, probable that the higher, diverging, terms are *de facto* negligible (see § **25**).

The scattering of fast electrons by electrons can easily be calculated from the matrix element (35). We shall not, however, discuss this here.†

A. PROCESSES OF FIRST ORDER IN e^2

11. Emission and absorption‡

The emission and absorption of light by an atom (or of γ-rays by a nucleus) can easily be understood in the light of the preceding theory. According to the considerations in § **7** and § **10** an atom and the radiation field form two quantum-mechanical systems with an interaction energy H' (§ **10** eqq. (2c) and (3)) which depends upon the coordinates of both systems. This interaction H', regarded as a small perturbation, will cause transitions of the unperturbed system (atom+radiation) in general consisting (i) of a transition of the atom from one quantum state to another one, and (ii) of an emission or absorption of light quanta.

† See Møller, loc. cit., and Mott and Massey, *Theory of Atomic Collisions*, Oxford, 1933, p. 264.

‡ The theory of emission and absorption of light in the form presented in this section was first developed by P. A. M. Dirac, *Proc. Roy. Soc.* **114** (1927), 243 and 710 (dispersion).

The radiation field has a continuous spectrum. If for instance a light quantum $\mathbf{k}$ ($k = \hbar\nu$) is emitted, we have the choice of assigning this quantum to any one of a very great number of radiation oscillators

$$\rho_k dk = \frac{k^2 dk d\Omega}{(2\pi\hbar c)^3}, \tag{1}$$

which all have the same frequency (within the interval dk), the same direction of propagation (within the element of the solid angle $d\Omega$), and the same polarization. Hence it follows, according to § 9, that the transition probability is proportional to the time (at least for sufficiently short times). Furthermore, the *energy of the unperturbed system (atom+radiation field)* is *conserved* for all transitions in which light quanta are emitted or absorbed.

The interaction between the atom and the radiation field can cause these radiative transitions even if, in the initial state, *no light quanta at all are present.* Supposing the atom to be excited in the initial state, then in the final state the number of light quanta will be increased from zero to some finite value. This process represents then a *spontaneous emission of light.* (Compare the corresponding classical considerations in § 6. 5, on the emission of light as an excitation of radiation oscillators.)

In § 10 we have seen that the radiation processes of the first order are those in which only *one light quantum* is emitted or absorbed. These transitions occur directly, without any intermediate states. The transition probabilities according to § 9 eqq. (42) and (43 a) are then given by the matrix elements of H' for the direct transition from the initial state to the final state.

For the theory of emission and absorption we may confine ourselves to the non-relativistic approximation. Even for heavy atoms the energy of the K-shell is still much smaller than mc^2, and the relativistic correction, though appreciable for Uranium and X-rays emitted in transitions to the K-shell, does not change the results principally. The same is true for the γ-rays emitted by a nucleus because the γ-rays are then emitted by heavy particles (protons, α-particles, neutrons) which move very slowly inside the nucleus. We may use therefore the interaction function H', § 10 eq. (3),

$$H' = -(e/\mu)(\mathbf{pA}). \tag{2}$$

The second term which is proportional to A^2 would give rise to transitions in which two quanta are involved; it can therefore be

neglected. The matrix elements of (2) for the emission or absorption of a quantum $\mathbf{k}_\lambda$ are given by (§ 10 eq. (12))

$$H'_{an_\lambda;\, bn_\lambda+1} = H'^{*}_{bn_\lambda+1;\, an_\lambda} = -\frac{e}{\mu}\sqrt{\left(\frac{2\pi\hbar^2c^2}{k_\lambda}\right)}\sqrt{(n_\lambda+1)}\int \psi_a^* p_e e^{i(\kappa_\lambda r)}\psi_b$$

$$= -\frac{e}{\mu}\sqrt{\left(\frac{2\pi\hbar^2c^2}{k_\lambda}\right)}\sqrt{(n_\lambda+1)}(p_e e^{i(\kappa_\lambda r)})_{ab}, \qquad (3)$$

say, where p_e denotes the component of $\mathbf{p}$ in the direction of polarization of $\mathbf{k}_\lambda$. For simplicity we confine ourselves to the case of one electron. If the atom contains several electrons the matrix element (3) must be written

$$e\int \psi_a^* p_e e^{i(\kappa_\lambda r)}\psi_b \to \sum_k e_k \int \psi_a^* p_{ek} e^{i(\kappa_\lambda r_k)}\psi_b. \qquad (3')$$

1. *Emission.* We calculate first the probability for the emission of light. We assume that there are two non-degenerate atomic states a, b with energies $E_a > E_b$. The conservation of energy states then that only light quanta with a frequency

$$k = \hbar\nu = E_a - E_b \qquad (4)$$

can be emitted. Equation (4) represents *Bohr's* well known *frequency relation.*

The transition probability per unit time is according to § 9 eq. (42)

$$w = \frac{2\pi}{\hbar}\rho_E |H'|^2, \qquad (5)$$

where ρ_E represents the number of final states with an energy between E and $E+dE$. For ρ_E in our case we have obviously to insert the number of radiation oscillators ρ_k since the light quantum can be placed in any of these oscillators. The energy of the final state is $k+E_b = E$ and therefore $dE = dk$, $\rho_E = \rho_k$. The formula (42) of § 9 (or our eq. (5)) has been obtained by taking the summation over all final states with the same energy, i.e. in our case over all radiation oscillators with the same physical properties. For $|H'|^2$ we have therefore to insert the average value of the square of the matrix element (3) over all these oscillators. Since this average value will then only depend upon the frequency, etc., but not upon the particular oscillator considered, we can omit the suffixes λ and replace n_λ by a quantity $\bar{n}_\nu$ denoting the *average number of light quanta per oscillator* having the frequency ν (and the direction of propagation $\mathbf{k}$, etc.).

Inserting equations (1) and (3) in (5) we obtain the transition

probability per unit time for the emission of a light quantum $\hbar\nu$ in a given direction:

$$w\,d\Omega = \frac{e^2}{\mu^2}\frac{\nu\,d\Omega}{2\pi\hbar c}|(p_e\,e^{i(\kappa\mathbf{r})})_{ab}|^2(\bar{n}_\nu+1). \tag{6}$$

In general one can assume that the wave-length of the light emitted, $1/\kappa$, is large compared with the dimensions of the atom. If E is the energy of the atom, the wave-length will be of the order of magnitude

$$\lambda \sim \hbar c/E. \tag{7}$$

On the other hand, the dimensions a of the atom will be determined roughly by an equation of the type

$$E \sim e^2/a \quad \text{or} \quad a \sim e^2/E. \tag{8}$$

Since $\hbar c/e^2 = 137$, λ will be large compared with a. We can then neglect the factor $\exp(i(\boldsymbol{\kappa}\mathbf{r}))$ in (6), since it is nearly constant in the region where ψ_a or ψ_b is different from zero. Putting $\mathbf{p}/\mu = \mathbf{v}/c$ and introducing the notation Θ for the angle between the direction of polarization and the vector $\mathbf{v}$, the transition probability becomes

$$w\,d\Omega = \frac{e^2\nu\,d\Omega}{2\pi\hbar c^3}|\mathbf{v}_{ab}|^2\cos^2\Theta(\bar{n}_\nu+1), \tag{9}$$

where $|\mathbf{v}_{ab}|^2 = v_{xab}^2+v_{yab}^2+v_{zab}^2$ and v_{xab} represents the matrix element of the x-component of v for the transition $a \to b$. Since, in the quantum theory, any matrix element ab is proportional to

$$\exp\{it(E_a-E_b)/\hbar\} = \exp(i\nu t)$$

we can also write $$v_{xab} = \dot{x}_{ab} = i\nu x_{ab}$$

and obtain $$w\,d\Omega = \frac{e^2\nu^3\,d\Omega}{2\pi\hbar c^3}|\mathbf{x}_{ab}|^2\cos^2\Theta(1+\bar{n}_\nu). \tag{10}$$

The probability of emission consists, according to (10), of two terms. The first term is independent of the intensity of radiation present before the emission. It gives rise to the *spontaneous emission* and is different from zero even if $\bar{n}_\nu = 0$. The second term, however, is proportional to the intensity of radiation $\bar{n}_\nu$ of frequency ν present before the emission process. This term gives rise to a certain *induced emission of radiation*. The existence of such an induced emission was first postulated by Einstein, who has shown that it is necessary to account for the thermal equilibrium in a gas emitting and absorbing radiation (see p. 108). It can be deduced from the classical

theory by means of the correspondence principle, since in § 5 we have seen that a light wave falling upon an oscillator not only causes an absorption of light but also, for certain phase differences between the oscillator and incident light wave, an emission of light. The analogous process in the quantum theory is given by the term proportional to $\bar{n}_\nu$.†

The total intensity radiated per unit time is obtained by multiplying (10) by $\hbar\nu$ and integrating over the angles. Taking first the summation over the directions of polarization we obtain, instead of $\cos^2\Theta$, $\sin^2\theta$, where θ represents the angle between the vector $\mathbf{x}$ (position of the electron related to the nucleus) and the direction of propagation $\mathbf{k}$. We obtain for the spontaneous emission:

$$S\,d\Omega = \frac{e^2\nu^4\,d\Omega}{2\pi c^3}|\mathbf{x}_{ab}|^2\sin^2\theta. \tag{11}$$

(11) gives the intensity emitted per unit time in the direction $\mathbf{k}$. The total intensity is given by integrating (11) over all angles

$$S = \frac{4}{3}\frac{e^2}{c^3}\nu^4|\mathbf{x}_{ab}|^2. \tag{12}$$

The formulae (11) and (12) are almost identical with the formulae obtained for an oscillator in the classical theory (§ 3 eqq. (23) and (25) or § 6 eq. (60)). We have only to replace the time average of the coordinate of the oscillator $\overline{x^2}$ by the *matrix element* of the same quantity for the transition $a \to b$

$$\overline{x^2} \to 2|\mathbf{x}_{ab}|^2. \tag{13}$$

(13) gives the well-known connexion between the classical quantities and the quantum theoretical quantities according to the correspondence principle.‡

If the atom contains several electrons, $\mathbf{x}_{ab}$ has to be equated to (according to (3'))

$$e\mathbf{x}_{ab} \to \sum_k e_k\mathbf{x}_{kab}. \tag{14}$$

(14) represents the total *dipole moment* of the atom, and corresponds exactly to the classical formula § 3 eq. (20) for an oscillator. The radiation given by (12) is *dipole radiation* (cf. subsection 3) with the same intensity as would be emitted by a classical oscillator with the

† Cf. Planck, *Wärmestrahlung*, Leipzig, 1923, p. 145 ff.

‡ The factor 2 in (13) is inserted because the matrix element x_{ab} corresponds to a single classical Fourier component $x_\nu \exp(i\nu t)$. For a classical oscillator with an amplitude $x_0\cos\nu t = \frac{1}{2}x_0[\exp(i\nu t)+\exp(-i\nu t)]$ we have two Fourier components with frequencies $+\nu$ and $-\nu$.

amplitude (13). To describe the corresponding radiation we can attribute to each transition $a \to b$ an oscillator with the frequency $\nu_{ab} = (E_a - E_b)/\hbar$ and an amplitude given by (13).

The order of magnitude of the transition probability per unit time will be according to (10), (7), (8) (putting $x_{ab} \sim a$)

$$w \sim \frac{e^2}{\hbar c^3} \nu^3 a^2 \sim \frac{1}{137} \left(\frac{\nu a}{c}\right)^2 \nu \sim \frac{\nu}{137^3}, \tag{15}$$

i.e. of the order 10^8 sec.$^{-1}$ for the optical region, 10^{11} for X-rays, and 10^{14} sec.$^{-1}$ for γ-rays. It is independent of the mass of the emitting particle, but not of course of the charge.

2. *Absorption.* The probability of the absorption of a light quantum can be obtained in the same way. We consider a light beam of the intensity $I_0(\nu)\, d\nu$ (energy per cm.^{2} sec.) coming from a given direction within an element of the solid angle $d\Omega$. The probability per unit time that one light quantum of this beam is absorbed while the atom jumps from b to a is again given by formula (5), ρ_E denoting now the number of initial states. The light quantum can be absorbed from any of the radiation oscillators. ρ_E is therefore again given by (1). The averaging over all radiation oscillators must be carried out in the same way as in subsection 1. If in the initial state the average number of quanta per oscillator is $\bar{n}_\nu$, $I_0(\nu)$ is given by

$$I_0(\nu)\, d\nu = \bar{n}_\nu \hbar\nu \cdot \frac{k^2\, dk d\Omega}{(2\pi\hbar c)^3} c = \bar{n}_\nu \frac{\nu^3\, d\Omega d\nu}{(2\pi)^3 c^2} \hbar. \tag{16}$$

The probability of absorption per unit time is again given by (3) and (5) (transition from $\bar{n}_\nu$ to $\bar{n}_\nu - 1$)

$$w\, d\Omega = \frac{e^2}{\mu^2} \frac{\nu\, d\Omega}{2\pi\hbar c} |(p_e e^{i(\mathfrak{k}\mathfrak{r})})_{ab}|^2 \bar{n}_\nu. \tag{17}$$

The probability of absorption is proportional to the intensity of the incident light beam as is to be expected. The coefficient is exactly the same as for the emission. The ratio of the two probabilities is therefore

$$\frac{w_{\text{emission}}}{w_{\text{absorption}}} = \frac{\bar{n}_\nu + 1}{\bar{n}_\nu}. \tag{18}$$

The ratio (18) is just that which is necessary to preserve the correct thermal equilibrium of the radiation with a gas. In a gas at the temperature T the number of atoms in the states a and b are given by $\exp(-E_a/kT)$ and $\exp(-E_b/kT)$. The condition for the equili-

brium of the gas with the radiation of frequency ν (and given direction of propagation, etc.) is

$$w_{\text{emission}}\, e^{-E_a/kT} = w_{\text{absorption}}\, e^{-E_b/kT}.$$

Inserting (18) and taking into account that $E_a - E_b = \hbar\nu$ we obtain Planck's formula:

$$\bar{n}_\nu = \frac{1}{e^{\hbar\nu/kT} - 1}.$$

From this deduction of Planck's law Einstein was the first to show the necessity for the existence of *induced emission* in the quantum theory.

For the same reason as in subsection 1 we can omit the factor $\exp i(\boldsymbol{\varkappa}\mathbf{r})$ in (17). Averaging over all orientations of the atom (i.e. over the directions of $\mathbf{x}$, $\overline{\cos^2\Theta} = \frac{1}{3}$) relative to the incident beam and introducing the primary intensity $I_0(\nu)$ (16) instead of $\bar{n}_\nu$, we obtain for the energy absorbed per unit time:

$$S = \frac{4\pi^2}{3}\frac{e^2}{\hbar c}\nu|\mathbf{x}_{ab}|^2 I_0(\nu). \tag{19}$$

This formula corresponds to the formula (19) § 5 obtained for the absorption of a classical oscillator. For a 3-dimensional oscillator the quantum theory gives just

$$|\mathbf{x}_{ab}|^2 = \frac{3\hbar}{2m\nu}$$

and hence the classical formula deduced in § 5:

$$S = 2\pi^2\frac{e^2}{mc} I_0(\nu).$$

If the levels a and b are *degenerate* with multiplicity, say g_a and g_b fold, then in order to obtain the total transition probability $a \to b$, we must take the sum over all final levels b and the average over all initial levels a. If we distinguish the g_a states of a by indices m_a, the matrix element for emission becomes (if $E_a > E_b$)

$$|\mathbf{x}_{ab}|^2 = \frac{1}{g_a}\sum_{m_a, m_b} |\mathbf{x}_{m_a m_b}|^2 \tag{20 a}$$

and for absorption

$$|\mathbf{x}_{ba}|^2 = \frac{1}{g_b}\sum_{m_a, m_b} |\mathbf{x}_{m_a m_b}|^2. \tag{20 b}$$

The matrix elements for emission and absorption differ then by the ratio of the statistical weights g:

$$|\mathbf{x}_{ab}|^2 = \frac{g_b}{g_a}|\mathbf{x}_{ba}|^2. \tag{21}$$

3. *Quadripole radiation.* In subsection 1 we have seen that the matrix element occurring in (3) which is responsible for the emission of light can in general be replaced by the matrix element $\mathbf{x}_{ab}$ of the dipole moment. For certain transitions $a \to b$, however, it may happen that the matrix elements of the dipole moment $\mathbf{x}_{ab}$ vanish. Those are classed as transitions forbidden by the selection rules. If, for a transition $a \to b$, $\mathbf{x}_{ab} = 0$, it is still possible that (3) does not vanish and that the transition occurs though in a higher approximation and with a smaller transition probability.

For a wave-length $\lambda \gg a$ we can expand the exponential function occurring in the matrix element of (3) in powers of the ratio a/λ:

$$e^{i(\boldsymbol{\kappa}\mathbf{r})} = 1 + i(\boldsymbol{\kappa}\mathbf{r}) + \ldots. \tag{22}$$

The matrix element (3) can then be developed in a similar way (inserting $i\nu x_e/c$ for p_e/μ and denoting the displacement of the electron by $\mathbf{x}$ instead of $\mathbf{r}$)

$$(x_e e^{i(\boldsymbol{\kappa}\mathbf{x})})_{ab} = x_{eab} + i\{x_e(\boldsymbol{\kappa}\mathbf{x})\}_{ab}. \tag{23}$$

If, for a forbidden transition, $\mathbf{x}_{ab} = 0$, the second term of (23) may still give a certain transition probability. The intensity is, according to (6), given by

$$S\,d\Omega = \frac{e^2\nu^4 d\Omega}{2\pi c^3}|\{\mathbf{x}(\boldsymbol{\kappa}\mathbf{x})\}_{ab}|^2\sin^2\theta. \tag{24}$$

The order of magnitude of (24) is smaller than for an allowed transition (11). Since $\kappa = 1/\lambda$ and $x \sim a$, the ratio of the intensity of a 'forbidden transition' to the intensity of an 'allowed transition' is of the order $(a/\lambda)^2$ (supposing, of course, that the matrix elements of (24) do not vanish).

The expansion (23) corresponds exactly to the expansion of the Hertzian vector $\mathbf{Z}$ in the classical theory (§ 3 eq. (22)). The first term represents the dipole moment. The second term for a harmonically vibrating system of electrons can be written:

$$\mathbf{Z}_2 = \nu \sum e_k \mathbf{x}_k \frac{(\mathbf{x}_k \mathbf{R})}{Rc} = \sum_k e_k \mathbf{x}_k (\boldsymbol{\kappa}\mathbf{x}_k). \tag{25}$$

($\mathbf{R}$ denotes the vector from the nucleus to the point of observation

$\mathbf{R}/R = \varkappa/\kappa$.) (25) is just the quantity occurring in (24). It represents the *quadripole moment* of the atom. Thus, the radiation given by the second term of (23) and by (24) represents the quadripole radiation of the atom. As shown in atomic physics it is emitted in all transitions in which the angular momentum $l\hbar$ of the atom jumps by 0 or $2\hbar$.

The expansion (23) can also be considered as an expansion by which the *retardation* between the different points of the atom is taken into account in successive degrees of approximation.

We see, that for the problem of emission and absorption of light, the quantum theory gives results which correspond in every detail to those of the classical theory, in the sense of Bohr's correspondence principle.

12. Theory of the natural line breadth

The theory of the emission of light developed in the preceding paragraph gives a complete account of the intensity of a spectral line emitted by an atom. This theory was based on a first order approximation in which the interaction between the atom and the radiation was considered as a small perturbation. The radiation emitted in an atomic transition had a *sharp frequency* which was given by Bohr's relation

$$\hbar\nu = E_a - E_b. \tag{1}$$

In the classical treatment of this problem we have seen, however, that the line emitted by an oscillator is not infinitely sharp. It has a certain *natural breadth* γ corresponding to an intensity distribution (§ 4 eq. (28))

$$I(\nu) = I_0 \frac{\gamma}{2\pi} \frac{1}{(\nu-\nu_0)^2+\gamma^2/4}, \tag{2}$$

where ν_0 is the frequency of the oscillator. This natural breadth is due to the *reaction force* of the emitted radiation on the oscillator (self-force of the electron). In the approximation, however, where the natural breadth is taken into account the reaction force can also be derived simply from the law of conservation of energy. In this approximation, therefore, it has no connexion with the problem of the structure of the electron as is the case in all higher approximations.

In the quantum-mechanical formalism the reaction of the radiation on the atom is included just as fully as in the classical theory. The perturbation theory used ensures that the conservation of energy shall be automatically fulfilled (see § 9 subsection 3). We may therefore

expect with some confidence that the quantum theory will also account for the natural breadth of a spectral line and moreover in such a way that the difficulties connected with the higher approximations of the self-force (cf. § **10** subsection 1, § **18** subsection 2) do not appear.

In fact, the smallness of the interaction H' between atom and radiation was not the only assumption made in our perturbation theory (§ 9). (This assumption is of course fundamental.) We have solved the general equations (32) § 9 only for a time t, which is small compared with the lifetime of the initial state, so that, up to the time t, the probability for a transition is very small. Only for this case was the transition probability proportional to t. It is clear that this assumption makes impossible a theory of the line breadth, because the latter is due just to the gradual decrease (in the classical theory) of the amplitude of the oscillator or, in the quantum theory, to the falling off of the probability of the atom being in its initial state.

Weisskopf and Wigner† have now given an improved solution of the equations of the perturbation theory which is valid also for times t comparable with the reciprocal of the transition probability and which gives, in a simple way, the formula for the line breadth which we require.

1. *Atom with two states.* We consider first the simple case of an atom with two states a, b only ($E_a > E_b$). We have to return to the general differential equations § 9 (32) for the probability amplitudes of the states of the unperturbed system (atom+radiation). It is sufficient to take only those states into account which can be reached directly from the initial state (first approximation in H') and for which the energy is at any rate *nearly* conserved. We assume that at the time $t = 0$ the atom is excited and that no light quantum is present. Then we may confine ourselves to the consideration of those states where the atom has jumped down to the lower state and one light quantum $\hbar\nu$ has been emitted with a frequency *nearly* equal to that given by (1). Denoting the probability amplitudes by $b_{a,0}$ and $b_{b,1_\lambda}$ the differential equations (32) § 9 become

$$-i\hbar\dot{b}_{a,0} = \sum_\lambda H'_{a0;b1_\lambda} b_{b1_\lambda}(t) e^{i(E_b - E_a + k_\lambda)t/\hbar}, \tag{3a}$$

$$-i\hbar\dot{b}_{b,1_\lambda} = H'_{b1_\lambda;a0} b_{a0}(t) e^{i(E_a - E_b - k_\lambda)t/\hbar}. \tag{3b}$$

† V. Weisskopf and E. Wigner, *Zs. f. Phys.* **63** (1930), 54; ibid. **65** (1930), 18. Compare also L. Rosenfeld, ibid. **71** (1931), 273; H. Casimir, ibid. **81** (1933), 496; F. Bloch, *Phys. Zs.* **29** (1928), 58.

The initial conditions are

$$b_{a0}(0) = 1, \qquad b_{b1_\lambda}(0) = 0. \tag{3c}$$

Since we wish to describe the way in which the probability amplitude b_{a0} has decreased after the time t, we may not insert in the right-hand side of (3a), (3b) the values (3c) for $t = 0$, as we have done in § 11. We try to solve the equations (3) putting

$$b_{a0}(t) = e^{-\gamma t/2}, \tag{4}$$

i.e. we assume that the *probability of finding the atom in the excited state decreases exponentially* with an *average lifetime* $1/\gamma$. (3c) is obviously satisfied by (4).

Inserting (4) in (3b) we obtain the differential equation

$$-i\hbar\dot{b}_{b,1_\lambda} = H'_{b1_\lambda;\,a0}e^{i(E_a-E_b-k_\lambda)t/\hbar}e^{-\gamma t/2} \tag{5}$$

with the solution

$$b_{b1_\lambda} = H'_{b1_\lambda;\,a0}\frac{e^{i(\nu_0-\nu_\lambda)t-\gamma t/2}-1}{\hbar(\nu_0-\nu_\lambda+i\gamma/2)}, \tag{6}$$

where we have put

$$E_a - E_b = k_0 = \hbar\nu_0. \tag{7}$$

Finally we have to satisfy equation (3a). Inserting (6) into (3a) we obtain

$$\frac{i\hbar\gamma}{2} = \sum_\lambda \frac{|H'|^2[1-e^{[i(\nu_\lambda-\nu_0)+\gamma/2]t}]}{\hbar(\nu_0-\nu_\lambda+i\gamma/2)}. \tag{8}$$

The summation over the radiation oscillators λ on the right-hand side can be replaced, just as in § 9, by an integration over the frequencies ν. The integral (8) has then a sharp maximum for $\nu = \nu_0$, supposing that γ is small compared with ν_0. This has of course to be assumed since $1/\gamma$ represents the lifetime of the initial state, which according to § 10, must always be large compared with the period of revolution of the electron. If $\rho_k\,dk\,d\Omega$ represents as before the number of radiation oscillators per unit volume with given physical properties (frequency, polarization, etc.), the right-hand side of (8) reduces to the well-known integral

$$\int f(\nu)\frac{1-e^{[i(\nu-\nu_0)+\gamma/2]t}}{\nu_0-\nu+i\gamma/2}\,d\nu = i\pi f(\nu_0) \tag{9}$$

with

$$f(\nu) = \int \rho_k|H'|^2\,d\Omega, \tag{10}$$

where $\int d\Omega$ denotes integration over all directions of propagation

and summation over both directions of polarization. Hence we obtain for γ:

$$\gamma = \frac{2\pi}{\hbar}\rho_k \int |H'|^2 \, d\Omega = w_{ab}. \tag{11}$$

According to § **11** eq. (5) γ *is just equal* to the *total spontaneous transition probability per unit time for emission* $a \to b$. This was to be expected since, according to (4), γ was defined as the reciprocal of the lifetime of the excited state.

The intensity distribution of the emitted line is given by the probability function for the final state b_{b1_λ}. After a time $t \gg 1/\gamma$, when the atom has certainly jumped down, the probability that a quantum $\hbar\nu_\lambda$ has been emitted is given by

$$|b_{b1_\lambda}^{(\infty)}|^2 = \frac{|H'|^2}{\hbar^2} \frac{1}{(\nu_\lambda - \nu_0)^2 + \gamma^2/4},$$

or, integrating over all directions of propagation, etc., according to (11),

$$I(\nu)\, d\nu = \hbar\nu\rho_k \, dk \int |b_{b1_\lambda}^{(\infty)}|^2 \, d\Omega = \frac{\gamma}{2\pi} \frac{\hbar\nu \, d\nu}{(\nu - \nu_0)^2 + \gamma^2/4}. \tag{12}$$

The total intensity is $\hbar\nu = I_0$. Formula (12) is therefore identical with the classical formula § 4 (28), the only difference being that γ now represents the transition probability per unit time as given by (11) instead of $2e^2\nu_0^2/3mc^3$.

In the quantum theory a spectral line has therefore the same intensity distribution as in the classical theory (shown in Fig. 2, page 35). *The breadth at half maximum is equal to the total transition probability per unit time.* The maximum intensity lies at the frequency ν_0 given by the energy difference of the two states of the atom (7).

The relation between the half-value breadth and the transition probability can be understood from Heisenberg's *uncertainty relation* for energy and time:

$$\Delta E \Delta t = \hbar, \tag{13}$$

which states that the energy of a system is only known with an accuracy ΔE if, for the measurement of the energy, a time Δt is available. In our case the excited state of the atom has a lifetime $1/\gamma$ due to the radiative transition probability. Therefore the energy of the excited state is defined only with an uncertainty $\Delta E = \hbar\gamma$, or the energy level E_a has a *breadth* $\Delta E_a \simeq \hbar\gamma$. The frequency of the emitted line will then have the same breadth $\Delta\nu \sim \gamma$, which is just our relation.

2. *Several atomic states.* The case where the atom has several states $a, b, c, \ldots$ is more complicated. It has also been treated by Weisskopf and Wigner (loc. cit.). The result cannot, however, in this case be unambiguously determined from the classical analogy. One would perhaps expect the intensity distribution of a line emitted in a transition $a \to b$ to be determined as before by equation (11), with a half-value breadth γ_{ab} equal to the transition probability $a \to b$. Such an assumption would be suggested especially by the correspondence between the quantum transition $a \to b$ and a classical oscillator of frequency $(E_a - E_b)/\hbar$, which was pointed out in § 11. But this conclusion is not in agreement with the above considerations about the uncertainty relation (13). The quantum theory leads to a rather different result.

If we denote the atomic levels in the order of their energies by $a_1, a_2, \ldots$, we can, according to (13), attribute to each *level*, a_i say, a certain *breadth* given by the sum of all *transition probabilities* from a_i to all lower levels:

$$\Delta E_{a_i}/\hbar \equiv \gamma_i = \sum_{j<i} w_{a_i a_j}, \tag{14}$$

where $w_{a_i a_j}$ represents the transition probability for the transition $a_i \to a_j$. The breadth of a certain *line*, $a_i \to a_k$ say, is then given by the *sum of the breadths of the two levels* a_i *and* a_k:

$$\gamma_{ik} = \gamma_i + \gamma_k. \tag{15}$$

The intensity distribution is again the classical one, equation (12), with $\gamma = \gamma_{ik}$.

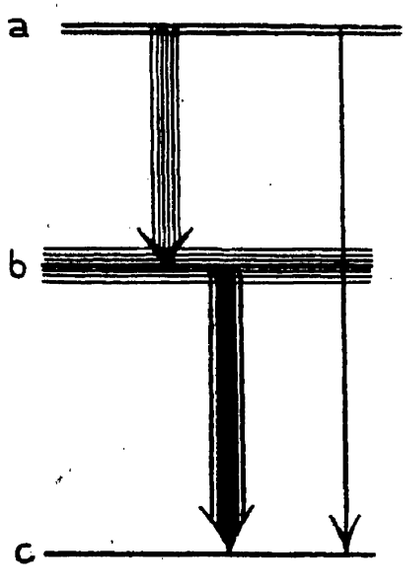

FIG. 4. 'Weak but broad' lines in the quantum theory of line breadth.

This result is quite different from that suggested by the correspondence principle and discussed above. There we concluded that the breadth of a line would be simply proportional to its intensity, as in the classical theory. This is, however, not true in the quantum theory. Here it may happen that even a weak line is rather broad. We may consider, for instance, a case as shown in Fig. 4 with three levels a, b, c. From the highest level a the transition probabilities are all small, and therefore a is a narrow level. From b, however, a strong line leads to the ground-level c (which is always infinitely sharp). Therefore b will be broad. According to equation (15)

the line $a \to b$ must also be broad although the transition probability is very small. The line $a \to c$, on the other hand, is narrow since it connects two narrow levels.

3. *Absorption.* The shape of the *absorption line* must be the same as the emission line if the incident light beam has a constant intensity in the region of the line breadth. This follows from general equilibrium considerations (Kirchhoff's law). If $I_0(\nu)\,d\nu = I_0(\nu_0)\,d\nu$ represents the intensity of the primary beam, the energy in the frequency range ν and $\nu+d\nu$ absorbed per unit time due to a transition $b \to a$ will be

$$S(\nu)\,d\nu = w_{ab}\frac{\pi^2c^2}{\nu_0^2}\,\frac{\gamma}{2\pi}\,\frac{I_0(\nu_0)\,d\nu}{(\nu-\nu_0)^2+\gamma^2/4}, \qquad \gamma = \gamma_a+\gamma_b, \tag{16}$$

where w_{ab} is the transition probability for the spontaneous emission $a \to b$. The factor in (16) has been determined so that the total energy absorbed per unit time is identical with formula (19) § **11**. If a and b are degenerate, (16) has to be multiplied by g_a/g_b (§ **11** (21)).

If we consider a layer of thickness Δx containing N atoms per cm.^{3} in the absorbing state b, we can define an *absorption coefficient per cm.* $\tau(\nu)$ for the primary light beam

$$\tau(\nu) = \frac{S(\nu)}{I_0(\nu_0)}N = Nw_{ab}\frac{\pi^2c^2}{\nu_0^2}\,\frac{\gamma}{2\pi}\,\frac{1}{(\nu-\nu_0)^2+\gamma^2/4}. \tag{17}$$

For frequencies at large distances from the maximum $(\nu-\nu_0)^2 \gg \gamma^2$ the absorption coefficient decreases with the square of the distance $\Delta\nu = \nu-\nu_0$, or in terms of the wave-length $2\pi\Delta\lambda = 2\pi c\Delta\nu/\nu^2$.

The ratio of the absorbed intensity to the primary intensity is then given by†

$$\tau(\lambda)\,\Delta x = N\,\Delta x\,w_{ab}\,\gamma\,\frac{\pi\lambda^6}{2c^2\,\Delta\lambda^2}. \tag{18}$$

(In (18) the wave-length is $2\pi\lambda$.)

4. *Other causes for the line breadth.* Besides the damping due to the emission of the radiation itself there are several other causes which actually broaden a line:

(a) In a gas of temperature T the atoms (mass M) move with velocities distributed according to Maxwell's law: $\exp(-v_x^2M/2kT)$. If we observe the light emitted in the x-direction, the line will be shifted because of the *Doppler effect* by an amount (§ 7 eq. (21), for $v \ll c$)

$$\Delta\nu = \nu_0\,v_x/c. \tag{19}$$

† (18) is valid only if $\tau(\lambda)\,\Delta x$ is small. For large values of $\tau(\lambda)\,\Delta x$ the left-hand side of (18) has to be replaced by $1-\exp\{-\tau(\lambda)\,\Delta x\}$.

Averaging, we obtain, therefore, a broad line with an intensity distribution

$$I(\nu)\,d\nu = \text{const.}\,d\nu\,e^{-Mc^2\Delta\nu^2/2\nu_0^2 kT} \tag{20a}$$

and a *breadth at half maximum*:

$$\delta = \nu_0\sqrt{\left(\frac{2kT}{Mc^2}\log 2\right)}. \tag{20b}$$

In general the Doppler breadth δ is much greater than the natural breadth γ. The intensity distribution is, however, exponential and therefore decreases rapidly with the distance from the maximum $\Delta\nu$ in contrast to the natural breadth which has a very large tail decreasing only with $1/\Delta\nu^2$. The intensity observed at large distances $\Delta\nu$ from the maximum (i.e. if $\Delta\nu \gg \delta$) is therefore entirely due to the natural breadth.

(*b*) In a gas of finite density an excited atom undergoes *collisions* with neighbouring atoms which may induce a transition to the ground-level. The effect of these collisions on the line breadth can be described in a way which corresponds exactly to the classical theory of Lorentz (§ 4 subsection 5): if the number of effective collisions per sec. is Γ the *lifetime* of the excited state a will be *shortened*. The total number of transitions per sec. (radiative+collisions) is now equal to $\gamma+\Gamma$. The breadth of the level a will therefore be

$$\Delta E_a/\hbar = \gamma+\Gamma. \tag{21}$$

The line emitted has the same intensity distribution as the natural line (16), the only difference being that γ has to be replaced by $\gamma+\Gamma$. (21) is identical with the classical formula deduced in § 4 subsection 5. For very low densities the effect of broadening by collisions becomes very small.

(*c*) Finally, the excited atom will have all kinds of interaction with the neighbouring atoms, which causes a shifting and splitting of the excited state (Stark effect, resonance coupling, etc.). These effects have been investigated by several authors. We shall not consider them here.† For very low densities they are likewise small.

5. *Experimental check.* For an experimental test of the quantum theory of the natural line breadth, in order to avoid all other kinds of broadening, one must take a gas of very low pressure and measure

† A review of the various effects of broadening of spectral lines with reference to all theoretical and experimental papers has been published by Weisskopf, *Phys. Zs.* **34** (1933), 1. Recent papers, H. Kuhn and F. London, *Phil. Mag.* **18** (1934), 983, and H. Kuhn, ibid. 987; V. Weisskopf, *The Observatory*, No. 713 (1933).

the intensity at large distances from the maximum. In the case of emission the intensity is then, however, very small. To obtain measurable intensities it is better to investigate the *absorption line* in very thick layers. The absorbed intensity, even at distances a thousand times the half-value breadth, can then become of the same order of magnitude as the primary intensity, while, for all smaller distances, the primary intensity is absorbed entirely.

The most conclusive test for our theory would be obtained from a case like that of Fig. 4, i.e. a line which is 'broad but weak'. Such a case is, for instance, just realized in the H_α line of the hydrogen atom. As an example we shall compute the shape of the H_α absorption line in all details. This can be done theoretically (for hydrogen all matrix elements can easily be worked out) without using any experimental data except the number of atoms in the absorbing layer.

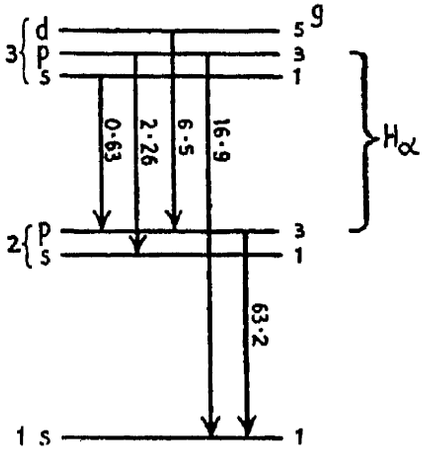

FIG. 5. Transition probabilities of the hydrogen lines (calculated) in emission. Units 10^7 sec.$^{-1}$.

The transition probabilities for the emission lines from the first two excited states, as calculated from § **11** eq. (12), are given in Fig. 5.† The three levels s, p, d of the three-quantum state are plotted separately, though actually they lie together. H_α consists of three components with practically the same frequency ($2\pi\lambda = 6563$ Å.U.). g denotes the statistical weights. The probability for the transition $3 \to 2$ is about ten times smaller than for the transition $2 \to 1$. The level 2 will be very broad and the line $3 \to 2$ will therefore be broad too, although the transition probability is small.

To obtain H_α as an absorption line a great number of atoms in the first excited state must be present. This condition is only realized in the *solar atmosphere*. The shape of H_α in the solar absorption spectrum has been measured by Unsöld‡ who has also compared his results with the theory.

To compute the shape of the line we require the total number of

† Cf. H. Bethe, *Hdb. d. Physik*, 2nd ed., XXIV (1), p. 444.

‡ A. Unsöld, *Zs. f. Astrophys.*, **2** (1931), 199. Measurements of the natural line breadth of other elements in the laboratory have been carried out by R. Minkowski, *Zs. f. Phys.* **36** (1926), 839; ibid. **55** (1929), 18; W. Schütz, ibid. **45** (1927), 30; ibid. **64** (1930), 682; *Naturw.* **20** (1932), 64; *Phys. Zs.* **36** (1935), 403; M. Weingeroff, ibid. **67** (1931), 679.

absorbing atoms in the layer. From other observations it has been found that in the solar atmosphere the number of hydrogen atoms in the first excited state per cm.^{2} of the total absorbing layer is

$$N\Delta x = 5\times 10^{15}\ \text{cm.}^{-2} \tag{22}$$

We determine first the breadth of all the levels concerned. From Fig. 5 we can take immediately (according to (14))

$$\left.\begin{aligned} \gamma_{3d} = 6{\cdot}5;\quad \gamma_{3p} = 19{\cdot}2;\quad \gamma_{3s} = 0{\cdot}63 \\ \gamma_{2p} = 63{\cdot}2;\quad \gamma_{2s} = 0 \end{aligned}\right\}\times 10^7\ \text{sec.}^{-1} \tag{23}$$

And hence the half-value breadth for the three components of H_α:

$$\left.\begin{aligned} \gamma_{3d,\,2p} &= 69{\cdot}7 \\ \gamma_{3p,\,2s} &= 19{\cdot}2 \\ \gamma_{3s,\,2p} &= 63{\cdot}8 \end{aligned}\right\}\times 10^7\ \text{sec.}^{-1} \tag{24}$$

Since the p and d levels are degenerate the transition probabilities for absorption differ from those for emission by the ratio of the statistical weights (according to § **11** (21) and Fig. 5):

$$\left.\begin{aligned} w_{2s,\,3p} &= 3w_{3p,\,2s} = 6{\cdot}8 \\ w_{2p,\,3d} &= \tfrac{5}{3}w_{3d,\,2p} = 10{\cdot}8 \\ w_{2p,\,3s} &= \tfrac{1}{3}w_{3s,\,2p} = 0{\cdot}2 \end{aligned}\right\}\times 10^7\ \text{sec.}^{-1} \tag{25}$$

To obtain the absorbed intensity, we have, according to (18), to calculate the product $w\gamma$ for each transition. The total intensity of H_α will then be given by the sum of the $w\gamma$'s for all three components of H_α. Taking into account the fact that the number of atoms in the $2s$-state is $N/4$ and in the $2p$-state $3N/4$, we obtain

$$\sum w\gamma = \tfrac{1}{4}w_{2s,\,3p}\gamma_{3p,\,2s} + \tfrac{3}{4}(w_{2p,\,3d}\gamma_{3d,\,2p} + w_{2p,\,3s}\gamma_{3s,\,2p}) = 607\times 10^{14}\ \text{sec.}^{-2} \tag{26}$$

Inserting (22) and (26) into our formula for the absorption coefficient, we obtain for the ratio of the absorbed intensity to the primary intensity at large distances from the maximum

$$\tau(\lambda)\Delta x = \frac{0{\cdot}27}{4\pi^2\,\Delta\lambda^2} \quad (2\pi\lambda = \text{wave-length in Å.U.}). \tag{27}$$

Thus, if $2\pi\,\Delta\lambda$ is of the order 1 Å.U., the absorbed intensity is of the order of magnitude of the primary intensity. The breadth at half maximum is only about 10^{-3} Å.U.

The theoretical shape of the absorption line of H_α is shown in Fig. 6. The centre of the line is absorbed entirely. For comparison

we have also plotted the shape according to the classical theory, i.e. if we replace γ by the actual transition probability (breadth proportional to intensity). The absorbed intensity would then be seven times smaller for all wave-lengths. The two experimental curves, for the solar edge and the solar centre, fit the *quantum theoretical curve definitely better than the classical one.* For the causes of the difference between the shape at the solar edge and at the solar centre and also for the origin of the observed remaining intensity in the middle of the line, cf. Unsöld (loc. cit.).†

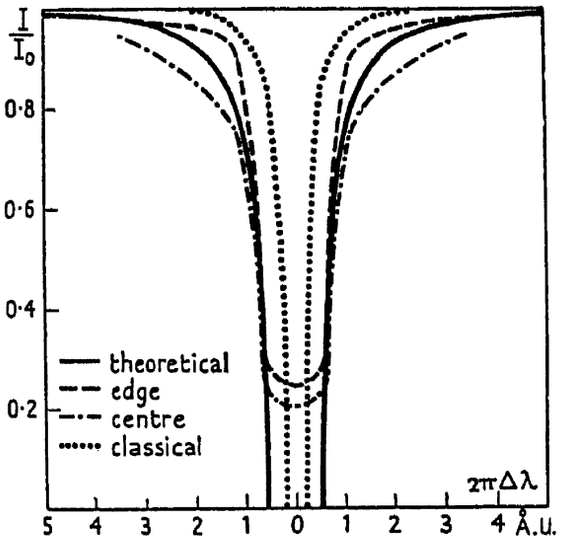

FIG. 6. Theoretical shape of the H_α absorption line for a layer of 5.10^{15} atoms per cm.^{2} in the two-quantum state. Observations on the solar spectrum at the edge and centre of the sun. The classical theory would give a much narrower line (dotted curve).

In the laboratory similar measurements have been carried out by W. Schütz (loc. cit.) with Ne lines. He was also able to verify in several examples the quantum theoretical result that the breadth of a line is determined by the breadths of the two levels involved but not by the transition probability of the line itself.

13. Photoelectric effect

Light of a given frequency ν according to § **11** can only be absorbed by an atom if ν is equal to one of the resonance frequencies $(E_a - E_b)/\hbar$ of the atom. If now the quantum energy $\hbar\nu$ of the incident light is greater than the ionization energy I of the atom, the electron is raised into a state of the continuous spectrum, and thus leaves the atom. In this case light of all frequencies can be absorbed, and the

† And other papers of the same author, *Zs. f. Astrophys.*, 1930–2.

absorption spectrum is *continuous*. The kinetic energy T of the electron after leaving the atom is determined by Einstein's equation

$$T = \hbar\nu - I. \tag{1}$$

The photoelectric effect plays in general an important part in the absorption of X-rays and γ-rays in matter (cf. Chapter V). It gives an appreciable absorption even if the primary energy is very much greater than the ionization energy of the atom. Since in this book we are especially interested in the *high* energy region, in our discussion of the photoelectric effect we shall keep in view the absorption of high-frequency radiation rather than of radiation in the optical region.

We shall, however, carry out the calculations for a very simple case only and shall quote the results obtained for other cases. Since a free electron cannot absorb light, we should expect the probability for the photoelectric absorption to be the greater the more strongly the electron is bound. We may therefore confine ourselves to the absorption by a *K-electron*. Furthermore we shall assume:

(*a*) The quantum energy of the incident light is to be large compared with the ionization energy I of the K-electron. For an atom with nuclear charge Z this condition, according to (1) and § 9 eq. (5), can be written as follows:

$$T = \frac{p^2}{2\mu} \gg I = \frac{Z^2\mu}{2\times 137^2}, \quad \text{or} \quad \xi = \frac{Ze^2}{\hbar v} \ll 1. \tag{2}$$

According to § 9 eq. (9), (2) is identical with the condition for the validity of Born's approximation. Therefore in the matrix elements we can replace the wave function of the electron in the continuous spectrum by a plane wave. Our results will not, of course, be valid in the neighbourhood of the absorption edge ($\hbar\nu \sim I$):

(*b*) The energy of the electron in the continuous spectrum, on the other hand, will be assumed small compared with mc^2, so that relativistic corrections are not important, i.e.

$$\hbar\nu \ll mc^2. \tag{3}$$

Actually the error will not be very large for energies up to about $0{\cdot}5mc^2$.

1. *Non-relativistic case, great distances from absorption edge.* The transition probability per unit time for photoelectric absorption is given by the general formula § 9 (42). Since the final state of the

electron belongs to the continuous spectrum, we can insert for ρ_E the number of quantum states per unit volume of the electron

$$\rho_E dE = \frac{pE\,dEd\Omega}{(2\pi\hbar c)^3} = \frac{\mu p\,dEd\Omega}{(2\pi\hbar c)^3}, \tag{4}$$

and may then assume that the primary radiation consists of a single light quantum $\hbar\nu$ only.†

For the matrix elements H' we have to insert the value § **10** eq. (12) for the absorption of a single quantum

$$H' = -\frac{e}{\mu}\sqrt{\left(\frac{2\pi\hbar^2c^2}{k}\right)}\int \psi_a^* p_e e^{-i(\mathbf{\kappa r})}\psi_b, \tag{5}$$

where p_e represents the component of the momentum in the direction of polarization of the primary quantum, ψ_a is the wave function of the electron in the K-shell, and ψ_b the wave function of the electron in the continuous spectrum with the momentum $\mathbf{p}$.

According to § **9** (3) and (8) we have

$$\psi_a = \frac{1}{\sqrt{(\pi a^3)}}e^{-r/a}, \qquad \psi_b = e^{i(\mathbf{pr})/\hbar c}, \qquad a = \frac{a_0}{Z}. \tag{6}$$

Introducing instead of a a quantity $\alpha = Z\hbar c/a_0$, which has the dimensions of an energy

$$\alpha = \frac{\hbar cZ}{a_0} = 2\frac{137}{Z}I = \sqrt{(2\mu I)} = \frac{Z}{137}\mu, \tag{7}$$

and a new vector $\mathbf{q}$ representing the momentum transferred to the atom

$$\mathbf{q} = \mathbf{p} - \mathbf{k}, \tag{8}$$

the integration of (5) yields

$$H' = -\frac{e}{\mu}p_e\sqrt{\left(\frac{\alpha^3}{\pi\hbar^3c^3}\right)}\sqrt{\left(\frac{2\pi\hbar^2c^2}{k}\right)}\frac{8\pi\alpha\hbar^3c^3}{(\alpha^2+q^2)^2}. \tag{9}$$

If we divide the transition probability per unit time by the intensity of the primary beam per cm.^{2}, i.e. for a single light quantum by the velocity of light c, we obtain a quantity of the dimensions of an area which we call the *differential cross-section* (differential, because it refers to an ejection of the electron into an element of the solid angle $d\Omega$).

† In the case of discrete absorption by an atom a transition probability per unit time exists only if the primary radiation consists of many quanta with a continuous intensity distribution. Cf. § **11** subsection 2.

According to (4) and (9) the differential cross-section is given by

$$d\phi = \frac{2\pi}{\hbar c}|H'|^2\rho_E\, d\Omega = \frac{32.137.r_0^2\mu p p_e^2\alpha^5\, d\Omega}{(\alpha^2+q^2)^4 k} \quad (r_0 = e^2/mc^2). \quad (10)$$

(10) gives the angular distribution of the ejected photoelectrons. Denoting by θ the angle between the direction of motion of the light quantum **k** and of the electron **p**, and by ϕ, the angle between the (**pk**)-plane and the plane formed by **k** and the direction of polarization **e**, i.e.

$$\theta = \angle\,(\mathbf{pk}),$$

$$\phi = \angle \text{ between } (\mathbf{pk})\text{-plane and } (\mathbf{ek})\text{-plane},$$

p_e and q can be expressed as follows

$$q^2 = p^2+k^2-2pk\cos\theta \quad (11\,a)$$

$$p_e = p\sin\theta\cos\phi. \quad (11\,b)$$

The denominator α^2+q^2 of (10) can be simplified. Since our formula (10) is in any case only correct for non-relativistic velocities we can make use of equation (3). According to (1) and (7) we have

$$k = \frac{\alpha^2+p^2}{2\mu} \ll \mu,$$

and hence†

$$\alpha^2+q^2 = \alpha^2+p^2+k^2-2pk\cos\theta = k(2\mu+k-2p\cos\theta)$$

$$\simeq 2\mu k(1-\beta\cos\theta), \qquad \beta = v/c = p/\mu. \quad (12)$$

Finally, α according to (7) and (2) is assumed to be small compared with p. We can therefore put $p^2 = 2k\mu$. Inserting again for α its value $Z\mu/137$, we obtain for the differential cross-section

$$d\phi = r_0^2\frac{Z^5}{137^4}\left(\frac{\mu}{k}\right)^{7/2}\frac{4\sqrt{2}\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^4}\, d\Omega. \qquad \text{N.R.} \quad (13)$$

Most of the photoelectrons are therefore emitted in the direction of polarization of the primary light quantum ($\theta = \frac{1}{2}\pi$, $\phi = 0$). In the direction of the light quantum **k** itself the intensity of photoelectric emission is zero (Fig. 7). The denominator of (13) gives, however, a slight preponderance in the forward direction which becomes more important for increasing velocities of the electron. In the relativistic case the maximum is strongly displaced in the forward direction.

The total cross-section for the ejection of photoelectrons in any direction may be obtained by integrating (13) over all solid angles. Here we may neglect the term $\beta\cos\theta$ in the denominator. Multi-

† The relativistic correction gives only terms in $(v/c)^2$. It is, therefore, justifiable to keep terms in v/c in (12) and (13).

plying by a factor 2 to allow for the fact that the K-shell contains two electrons, the cross-section ϕ_K for the photo effect of the K-shell becomes (expressed in terms of the ratio k/I and of k/μ)

$$\phi_K = \phi_0 \frac{Z^5}{137^4} 4\sqrt{2} \left(\frac{\mu}{k}\right)^{7/2} = \phi_0 \, 64 \frac{137^3}{Z^2} \left(\frac{I}{k}\right)^{7/2}, \quad \text{N.R.} \quad (14)$$

where $\phi_0 = 8\pi r_0^2/3$ is the cross-section for the Thomson scattering (§ 5 eq. (5)), which we may use as a convenient unit.

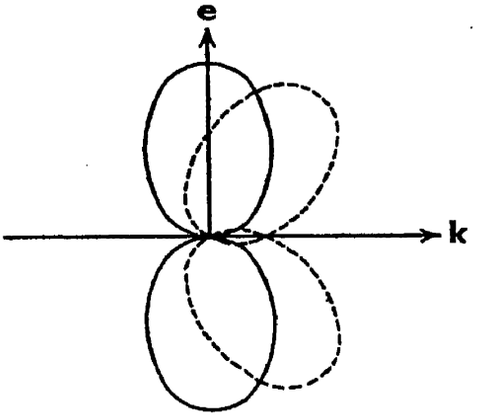

FIG. 7. Intensity distribution of photoelectrons with velocity $v \ll c$ and with velocity $v = \frac{1}{3}c$ (dotted curve). **e** = direction of polarization of the primary quantum **k**. If **k** increases, the maximum is displaced in the forward direction.

From (14) we obtain for the K-shell the *absorption coefficient* τ_K per cm., for radiation of frequency ν by multiplying by the number N of atoms per cm.^{3} (see § 22)

$$\tau_K = N\phi_0 \frac{Z^5}{137^4} 4\sqrt{2} \left(\frac{\mu}{k}\right)^{7/2} \quad \text{N.R.} \quad (14')$$

The absorption coefficient decreases rapidly, with the 7/2th power of the frequency. This, however, is only true as long as our assumptions (2) and (3) are justified.

In Fig. 8 we have plotted $\log_{10} \phi_K$ (in units of ϕ_0) for C, Al, Cu, Sn, Pb on a logarithmic scale in order to cover a large frequency range. Formula (14) gives in this diagram a straight line with a tangent $-3{\cdot}5$ (dotted curves for $\hbar\nu < 0{\cdot}5mc^2$). The deviations from the straight lines are due to the two corrections which will be considered in subsections 2 and 3.

2. *Neighbourhood of absorption edge.* Born's approximation is no longer valid if $\hbar\nu$ is so small that the energy of the ejected electron is of the same order of magnitude as the ionization energy I. In this case the exact wave functions of the continuous spectrum must be

used instead of plane waves (cf. § 9 subsection 1). Except for the very heaviest elements the non-relativistic approximation is sufficient.

The matrix elements (5) with the correct wave functions of the continuous spectrum have been computed by Stobbe.† Since the integration is rather tedious we quote only the result. The total cross-

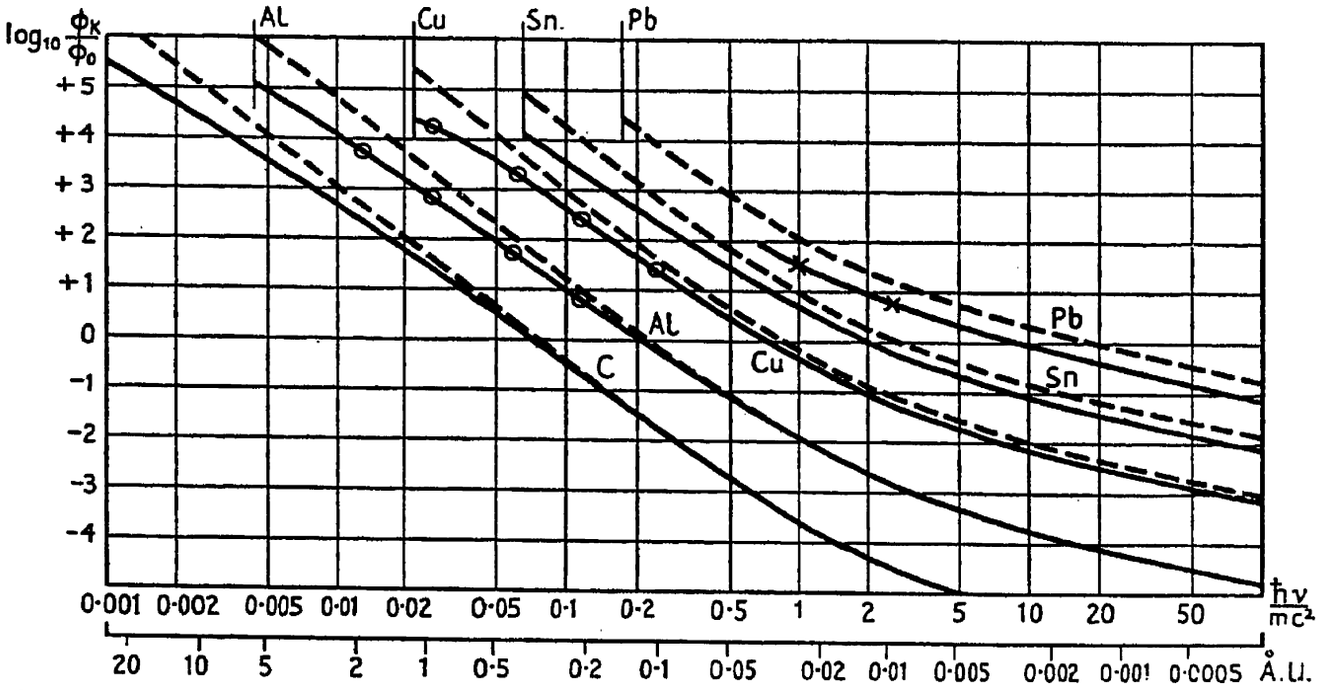

FIG. 8. $\log_{10}$ of the cross-section for photoelectric absorption in the K-shell for C, Al, Cu, Sn, Pb on a logarithmic scale. Above are shown the K-absorption edges. (For C the absorption edge lies just outside the diagram.) The dotted curves (straight lines for $h\nu < 0{\cdot}5mc^2$) are calculated with Born's approximation (formulae (14) and (17)). The deviation from the straight lines for $h\nu > 0{\cdot}5mc^2$ is due to relativistic effects. The full-drawn curves are exact. They are interpolated from formula (16) and the exact numerical calculations given in Table II. The circles refer to measurements of Allen (loc. cit.), the crosses to those of Gray (loc. cit., p. 126).

section ϕ_K, eq. (14), has to be multiplied by a factor

$$f(\xi) = 2\pi\sqrt{\left(\frac{I}{k}\right)}\frac{e^{-4\xi\,\mathrm{arccot}\,\xi}}{1-e^{-2\pi\xi}}, \qquad \xi = \sqrt{\left(\frac{I}{k-I}\right)} = \frac{Ze^2}{\hbar v}, \tag{15}$$

thus

$$\frac{\phi_K}{\phi_0} = 128\pi\frac{137^3}{Z^2}\left(\frac{I}{k}\right)^4\frac{e^{-4\xi\,\mathrm{arccot}\,\xi}}{1-e^{-2\pi\xi}}. \qquad \text{N.R.} \tag{16}$$

ξ^2 represents the ratio of the ionization energy to the kinetic energy of the ejected electron. The factor $f(\xi)$ diminishes the cross-section in the immediate neighbourhood of the K-absorption edge ($\xi \to \infty$) by a factor $2\pi\exp(-4) = 0{\cdot}12$. Even for a distance of fifty times the ionization energy from the absorption edge $f(\xi)$ is still only 0·66.

The character of these deviations due to the factor (15) can be

† M. Stobbe, *Ann. d. Phys.* 7 (1930), 661.

seen from Fig. 8. The correct curves approach rather slowly the straight lines calculated with Born's approximation. For Cu and Al some results of measurements by Allen† have been plotted. The agreement is actually much better than can be seen from the diagram in which only $\log_{10}\phi_K$ is plotted. (For a detailed comparison with the experiments see Stobbe, loc. cit.).

For the soft X-ray region, however, one has to keep in mind that Fig. 8 gives only the absorption coefficient of the K-shell. The outer shells, of course, also make some contributions which have to be considered, especially if $\hbar\nu$ is smaller than the K-absorption edge (see subsection 3).

3. *Relativistic region.* If, on the other hand, the energy of the primary quantum is of the order mc^2 or larger, the *relativistic* wave functions for ψ_K and ψ_p must be used. Using Born's approximation which can be applied for light elements the matrix elements have been worked out by Sauter.‡ We quote the result only:

$$\frac{\phi_K}{\phi_0}=\frac{3}{2}\frac{Z^5}{137^4}\left(\frac{\mu}{k}\right)^5(\gamma^2-1)^{\frac{3}{2}}\left[\frac{4}{3}+\frac{\gamma(\gamma-2)}{\gamma+1}\left(1-\frac{1}{2\gamma\sqrt{(\gamma^2-1)}}\log\frac{\gamma+\sqrt{(\gamma^2-1)}}{\gamma-\sqrt{(\gamma^2-1)}}\right)\right]. \tag{17}$$

$$\gamma=\frac{1}{\sqrt{(1-v^2/c^2)}}=\frac{k+\mu}{\mu}. \tag{17'}$$

γ represents the ratio of the total energy (kinetic energy$+mc^2$) of the electron to the rest energy. For $\gamma\to 1$ formula (17) goes over into our non-relativistic formula (14).

For extremely high energies, $k\gg\mu$, (17) becomes

$$\frac{\phi_K}{\phi_0}=\frac{3}{2}\frac{Z^5}{137^4}\frac{\mu}{k}. \qquad \text{E.R.} \quad (18)$$

ϕ_K decreases in the relativistic region more slowly than in the non-relativistic one, for $k\gg\mu$ only with μ/k (instead of $(\mu/k)^{7/2}$ for $k\ll\mu$).

The curves in Fig. 8, therefore, begin to turn up for $k\sim\mu$ and finally tend to straight lines inclined to the axis with a tangent -1. The slower falling off of ϕ_K in the relativistic region has the effect that even for $k\sim 10mc^2$ the photo effect of heavy elements contributes an appreciable part to the total absorption (cf. § **16** and § **22**).

† S. J. M. Allen, *Phys. Rev.* **27** (1926), 266; **28** (1926), 907. From the measured absorption coefficient we have subtracted the part which is due to scattering (cf. § **16**).

‡ F. Sauter, *Ann. d. Phys.* **9** (1931), 217; ibid. **11** (1931), 454.

Finally, we give in Table I the values of ϕ_K in units $\phi_0 Z^5/137^4$ for the region where, at least for light elements, the factor (15) is not appreciable. In the units chosen ϕ_K depends only on the ratio k/μ but not on Z.

TABLE I

Theoretical values of $\phi_K 137^4/\phi_0 Z^5$ *(eq. (17)). (Born's approximation)*

k/μ	0·1	0·25	0·5	1	2	3
$\frac{\phi_K 137^4}{\phi_0 Z^5}$	$1{\cdot}79\times10^4$	794	81	10·4	2·04	0·97
k/μ	5	10	20	50	100	
$\frac{\phi_K 137^4}{\phi_0 Z^5}$	0·45	0·185	$8{\cdot}36\times10^{-2}$	$3{\cdot}13\times10^{-2}$	$1{\cdot}54\times10^{-2}$	

For heavy elements Sauter's formula breaks down. Hulme, Fowler, and others† have made exact numerical calculations of ϕ_K for a few elements and for two values of k in the range where relativistic effects are important. They found the following values for ϕ_K instead of those given in Table I.

TABLE II

Exact theoretical values for $\phi_K 137^4/\phi_0 Z^5$

k/μ	*Fe*	*Sn*	*Pb*‡
0·69	18	12·2	7·9
2·2	1·05	0·80	0·60

Finally, Hall§ has deduced a formula which is valid for all Z and $k \gg \mu$. His derivation is however open to some criticism.† He obtains

$$\frac{\phi_K}{\phi_0} = \frac{3}{2}\frac{Z^5}{137^4}\frac{\mu}{k}e^{-\pi\alpha+2\alpha^2(1-\log\alpha)} \qquad \left(\alpha = \frac{Z}{137}\right). \quad \text{E.R.} \quad (19)$$

At very high energies (19) differs from the formula (18), obtained by using Born's approximation, by a factor 2·2 for Pb and 1·5 for Cu.

In Fig. 8 the full curves are interpolated from the values of Table II and those given by equations (16) and (19). ϕ_K has been determined experimentally for lead by Gray.‖ His results fit excellently the theoretical curve.

To obtain the *absorption coefficient* per cm. one has to multiply the

† H. R. Hulme, J. McDougall, R. A. Buckingham, and R. H. Fowler, *Proc. Roy. Soc.* **149** (1935), 131.

‡ The authors quoted give the values for $Z = 84$. We have obtained the value for $Z = 82$ by interpolation.

§ H. Hall, *Phys. Rev.* **45** (1934), 620.

‖ L. H. Gray, *Proc. Camb. Phil. Soc.* **27** (1931), 103.

values given in Tables I and II by $N\phi_0 Z^5/137^4$. The values of the latter quantity are given in the appendix for several elements.

The above calculations, however, represent only the absorption coefficient for the K-shell. To obtain a rough estimate of the contribution of the higher shells, we may use the experimental result that about 80 per cent. of the total photoelectric absorption at high energies ($\sim mc^2$) is due to the K-shell.† In Fig. 13, p. 160, and Fig. 21, p. 216, we show, therefore, the theoretical values obtained in this section multiplied by a factor 5/4 to give the total photoelectric absorption.

4. *Internal conversion.* This is a special type of photoelectric absorption. If a nucleus emits γ-rays, it may happen that a γ-quantum is absorbed by a K-electron of the same atom. If b denotes the number of electrons ejected per unit time and g the number of γ-quanta which leave the atom without being absorbed, the ratio

$$\alpha = b/(g+b) \tag{20}$$

is called the internal conversion coefficient. α has been calculated in several papers by Hulme, Taylor, Mott, and Fisk‡ in a very satisfactory way. We shall not, however, enter into a detailed discussion of these rather lengthy calculations but shall only mention some of the points which have to be taken into account.

In this case it is convenient to expand the field radiated by the nucleus, not in a series of plane waves as hitherto in this book, but in a series of spherical waves. The eigenwaves obtained in this way can be classified so as to represent the field emitted by an electric *dipole*, *quadripole*, etc. (compare § 7, p. 62). The field can be quantized in exactly the same way as in our usual expansion in a series of plane waves. In analogy to the radiation emitted by an atom we may assume that the field emitted in a particular nuclear transition is a *pure* dipole or quadripole field.

Denoting by ϕ_λ, $\mathbf{A}_\lambda$ the potentials of such an eigenwave representing a dipole or quadripole field with the frequency ν, and including also the static field of the dipole (quadripole), the probability per

† Rutherford, Chadwick, and Ellis, *Radiations from Radioactive Substances*, p. 464.

‡ H. R. Hulme, *Proc. Roy. Soc.* **138** (1932), 643; H. M. Taylor and N. F. Mott, ibid. **138** (1932), 665; **142** (1933), 215; N. F. Mott, *Ann. Inst. Henri Poincaré*, 1933; C. D. Ellis and N. F. Mott, *Proc. Roy. Soc.* **139** (1933), 369; J. B. Fisk, ibid. **143** (1934), 674; J. B. Fisk and H. M. Taylor, ibid. **146** (1934), 178. The last two references contain the most complete numerical values. J. C. Jaegar and H. R. Hulme, *Proc. Roy. Soc.* **148** (1935), 708, have calculated the internal conversion coefficient with production of pairs (compare § 20).

unit time for the ejection of a K-electron from the state ψ_K to a state with the momentum $\mathbf{p}$ (element of solid angle $d\Omega$) is given by the well-known formula (§ **9** eq. (42), § **10** eq. (9))

$$b = \frac{2\pi}{\hbar} e^2 \left| \int \psi_K^* \{\phi_\lambda + (\boldsymbol{\alpha} \mathbf{A}_\lambda)\} \psi_{\mathbf{p}} \right|^2 \frac{\hbar}{2\nu} \frac{pE\, d\Omega}{(2\pi\hbar c)^3}. \tag{21}$$

For radioactive atoms and for soft γ-rays the wave-length is of the same order of magnitude as the radius of the K-shell. In the matrix element (21) the retardation will therefore play an essential role. According to former considerations (compare § **10** subsection 5) we can—formally—distinguish between two parts of the matrix element.

(1) The direct interaction between the K-electron and the nuclear particles. By a suitable normalization this part is chiefly contained in ϕ_λ.

(2) The effect of retardation, which can be considered as an emission and reabsorption of a light quantum.

If the first part only existed, the internal conversion would have to be considered as an Auger effect rather than as a photoelectric absorption. The first part will be relatively larger, the larger is the ratio of the wave-length to the radius of the K-shell.

To calculate the ratio b/g, which is the only quantity which can be measured directly, we must know g. If the internal conversion consisted of a photoelectric absorption of light quanta (i.e. if the second part of the matrix element only were effective), g would simply be given by the total transition probability w_0 of the nucleus in the absence of the outer electrons, $g = w_0 - b$. This, however, is not the case. By the first part of the interaction, the total transition probability of the nucleus is larger than in the case when no electrons are present. Thus we have to put

$$b = b_0 + b', \quad w = w_0 + b' = b + g, \tag{22}$$

where b' denotes that part of b which is due to the direct interaction of the K-electron and the nucleus (Auger effect). It has in fact been calculated by Taylor and Mott (loc. cit.), that for soft γ-rays b_0 is very small ($b \simeq b'$), so that the presence of the K-electrons hardly diminishes the total number of γ-quanta leaving the atom ($g \simeq w_0$).

To obtain α, b and b' must be computed separately. This has been done in the papers quoted above. We obtain two different curves for the internal conversion coefficient α as a function of the frequency corresponding to a pure dipole and quadripole radiation respectively.

For long waves the quadripole field is larger than the dipole field within the K-shell. Thus the quadripole radiation will give a higher internal conversion coefficient than the dipole radiation. For γ-rays of 200,000 volts energy α is about three times larger for quadripole radiation than for dipole radiation.

It has been shown (loc. cit.) that the experimental points lie fairly well on either of these two curves. There have been found, however, some discrepancies of $\sim$ 40 per cent. which may possibly be due to the fact that the radiation emitted by the nucleus in a particular transition is not a pure dipole or quadripole field but a mixture of both (containing perhaps magnetic dipole or higher fields).

The values of α range between about 0·002 for $\hbar\nu \sim mc^2$ to 0·25 for $\hbar\nu \sim 0{\cdot}5mc^2$ (quadripole radiation). For dipole radiation the values are roughly three times smaller.

For other details we must refer to the papers quoted above.

B. PROCESSES OF SECOND ORDER $\sim e^4$

In part A of this chapter we have investigated the radiation processes of the first order which are connected with the emission or absorption of a single light quantum. The processes of the second order are those in which two light quanta are involved. Their transition probabilities will be proportional to e^4. Of these the *scattering of light* is far the most important, since the simultaneous emission or absorption of two quanta is in all cases negligible. The scattering process consists in the absorption of a primary light quantum $\mathbf{k}_0$ and the simultaneous emission of a secondary quantum $\mathbf{k}$. The scattering electron (atom) may be left in either its initial state (coherent scattering), or, as in the Raman effect, in some other state.

The general character of the scattering processes depends on whether the energy of the primary quantum k_0 is of the same order of magnitude as the binding energy of the electron in the atom, or large compared with the binding energy. In the latter case the electron can be considered as free. The scattering of free electrons will be discussed in detail in § **16**.

14. Dispersion and Raman effect

We consider first the case where k_0 is of the same order of magnitude as the potential energy of the electron. This is roughly the region of visible light up to frequencies of soft X-rays. The same

simplifications can then be made as in the theory of emission. We can neglect all relativistic corrections. Furthermore we can in general assume that the wave-length of the primary quantum λ_0 and of the scattered quantum λ are both large compared with the dimensions of the atom.

In this paragraph we denote the states of the atom by n_i, in particular the initial state will be denoted by n_0, the final state by n (energies E_i, E_0, E). The conservation of energy states that the frequency of the scattered quantum k differs from k_0 by the energy difference of the atom

$$k - k_0 = E_0 - E. \tag{1}$$

In the case of *coherent* scattering (compare subsection 2) the atom is left in the same state $n_0 = n$. The frequency of the scattered quantum k is then the same as that of the primary quantum k_0. The case $E_0 \neq E$ represents the Raman effect.

On the other hand the momentum will not in general be conserved in the interaction of light with a bound electron.

1. *The dispersion formula.* According to § **10** the non-relativistic interaction between an electron and the radiation is given by

$$H' = -\frac{e}{\mu}(\mathbf{pA}) + \frac{e^2}{2\mu} A^2 = H_1' + H_2'. \tag{2}$$

In contrast to the theory of emission, here we must not neglect the second term, which is $\sim A^2$. As has been shown in § **10** this is just a second-order term having matrix elements for direct transitions in which the total number of light quanta changes by two.

The matrix elements of H_2' are given in § **10** eq. (13). In our case, i.e. for the transition $E_0, k_0 \to E, k$, we have

$$H_2' = \frac{e^2}{\mu} \frac{2\pi\hbar^2 c^2}{\sqrt{(k_0 k)}} \int \psi_{n_0}^* e^{i(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0, \mathbf{r})} \psi_n (\mathbf{e}_0 \mathbf{e}), \tag{3}$$

where $\mathbf{e}_0$, $\mathbf{e}$ represent unit vectors in the direction of polarization of the two quanta $\mathbf{k}_0, \mathbf{k}$. $\boldsymbol{\varkappa}_0$ and $\boldsymbol{\varkappa}$ are vectors of the length $1/\lambda$. ($\boldsymbol{\varkappa} = \mathbf{k}/\hbar c$.)

If the wave-lengths of $\mathbf{k}_0$ and $\mathbf{k}$ are large compared with the dimensions of the atom, the exponential function can be regarded as constant in the integral (3). The matrix element is then obviously only different from zero if $n_0 = n$, i.e. in the case of coherent scattering

$$H_2' = \frac{e^2}{\mu} \frac{2\pi\hbar^2 c^2}{\sqrt{(k_0 k)}} e^{i(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0, \mathbf{X})} \delta_{n_0 n} (\mathbf{e}_0 \mathbf{e}), \tag{4}$$

where $\mathbf{X}$ represents a vector indicating the position of the atom.

The first term H_1' of equation (2) is a first-order term. It can only cause transitions involving two light quanta through the agency of *intermediate states* which differ from the initial and final states by having only one light quantum emitted or absorbed. In our case there are obviously two kinds of such intermediate states differing in the order in which the two processes, emission of $\mathbf{k}$ and absorption of $\mathbf{k}_0$, take place.

I. $\mathbf{k}_0$ is absorbed first, therefore no light quantum is present. In the transition to the final state $\mathbf{k}$ is emitted.

II. $\mathbf{k}$ is emitted first. Therefore both light quanta $\mathbf{k}_0$ and $\mathbf{k}$ are present. In the transition to the final state $\mathbf{k}_0$ is absorbed.

In both possible intermediate states the atom may be excited in any quantum state n_i, since for the transitions to or from an intermediate state energy need not be conserved.

Denoting the initial state by A, the final state by F, and the two possible intermediate states by I and II, the matrix elements of the first term of H_1' for the transitions from A to I and II and then to F are given (according to § **10** eq. (12), dropping the dash and the index 1) by:

$$\begin{aligned} H_{AI} &= \\ H_{AII} &= \\ H_{IF} &= \\ H_{IIF} &= \end{aligned} \quad -\frac{e}{\mu}\sqrt{(2\pi\hbar^2c^2)}\begin{cases} \dfrac{1}{\sqrt{k_0}}\displaystyle\int \psi_{n_0}^* p_0\, e^{-i(\kappa_0 r)}\psi_{n_i} \\ \dfrac{1}{\sqrt{k}}\displaystyle\int \psi_{n_0}^* p e^{i(\kappa r)}\psi_{n_i} \\ \dfrac{1}{\sqrt{k}}\displaystyle\int \psi_{n_i}^* p e^{i(\kappa r)}\psi_{n} \\ \dfrac{1}{\sqrt{k_0}}\displaystyle\int \psi_{n_i}^* p_0\, e^{-i(\kappa_0 r)}\psi_{n}, \end{cases} \tag{5}$$

where p_0 and p denote simply the components of $\mathbf{p}$ in the direction of polarization of the two light quanta $\mathbf{k}_0$ and $\mathbf{k}$, respectively.

The energy differences of the initial state and the intermediate states are

$$E_A - E_I = E_0 + k_0 - E_i, \qquad E_A - E_{II} = E_0 - E_i - k. \tag{6}$$

The matrix elements of H_1' for the transition $A \to F$ are given by the general formula § **9** eq. (43 b). The summation has to be carried out over all intermediate states.

Assuming again that the wave-lengths of $\mathbf{k}_0$ and $\mathbf{k}$ are large

compared with the dimensions of the atom, we obtain in our case

$$H_1' = \frac{e^2}{\mu^2}\frac{2\pi\hbar^2 c^2}{\sqrt{(k_0 k)}} e^{i(\mathbf{\kappa}-\mathbf{\kappa}_0,\mathbf{x})} \sum_i \left(\frac{p_{0n_0}p_{n_i n}}{E_0 - E_i + k_0} + \frac{p_{n_0 n_i}p_{0n_i n}}{E_0 - E_i - k}\right), \dagger \qquad (7)$$

where $p_{0n_0 n_i}$ denotes the matrix element of p_0 for the transition $n_0 \to n_i$.

The total matrix element of H' is the sum of the matrix elements arising from the two terms H_1' and H_2'. Collecting our formulae (4) and (7) we obtain

$$H' = \frac{e^2}{\mu}\frac{2\pi\hbar^2 c^2}{\sqrt{(k_0 k)}} e^{i(\mathbf{\kappa}-\mathbf{\kappa}_0,\mathbf{x})}\left[\frac{1}{\mu}\sum_i \left(\frac{p_{0n_0}p_{n_i n}}{E_0 - E_i + k_0} + \frac{p_{n_0 n_i}p_{0n_i n}}{E_0 - E_i - k}\right) + \delta_{n_0 n}\cos\Theta\right], \qquad (8)$$

where Θ represents the angle between the directions of polarization of $\mathbf{k}_0$ and $\mathbf{k}$.

The transition probability per unit time is given according to § 9 (42) by

$$w = \frac{2\pi}{\hbar}|H'|^2 \rho_E, \qquad (9)$$

where ρ_E represents the number of final states per unit volume and per energy interval dE.

In our case ρ_E is the number of radiation oscillators per unit volume in which the scattered quantum $\mathbf{k}$ can be placed, or

$$\rho_E = \rho_k = \frac{k^2\, d\Omega}{(2\pi\hbar c)^3}. \qquad (10)$$

Dividing (9) by the intensity of the primary beam, i.e. for one light quantum by the velocity of light, we obtain the differential cross-section for the scattering of a light quantum $\mathbf{k}$ into an element of solid angle $d\Omega$ and with a given polarization

$$d\phi = r_0^2 \frac{k}{k_0} d\Omega \left[\frac{1}{\mu}\sum_i \left(\frac{p_{0n_0}p_{n_i n}}{E_0 - E_i + k_0} + \frac{p_{n_0 n_i}p_{0n_i n}}{E_0 - E_i - k}\right) + \delta_{nn_0}\cos\Theta\right]^2. \qquad (11)$$

This formula is of course not valid in the case of resonance, i.e. if $k_0 \simeq E_i - E_0$ (see § 15).

In the case of coherent scattering we obtain the well-known dispersion formula

$$d\phi = r_0^2\, d\Omega \left[\frac{1}{\mu}\sum_i \left(\frac{p_{0n_0}p_{n_i n_0}}{E_0 - E_i + k_0} + \frac{p_{n_0 n_i}p_{0n_i n_0}}{E_0 - E_i - k_0}\right) + \cos\Theta\right]^2. \qquad (12)$$

† The summation $\sum_i$ has also to be carried out over the states of the continuous spectrum of the atom. The contribution of the latter, however, is not very large for the optical region.

Equation (12) was first obtained by Kramers and Heisenberg† by an application of the correspondence principle to the classical theory. In fact, the first term of (12) corresponds exactly to the classical formula § 5 (11), if we neglect the damping γ and attribute to each quantum transition of the atom a classical oscillator with the frequency $(E_i - E_0)/\hbar$, and an 'oscillator strength' which is proportional to $p_{0n_0 n_i} p_{n_i n_0}$.‡ The existence of the second term $\cos\Theta$ in the dispersion formula was first shown by Waller.‖ It is identical with the formula describing the scattering of a free electron (§ 5 (4)).†† If $k_0 \gg E_i - E_0$ (but λ_0 is still large compared with the dimensions of the atom), the first term of (12) is small and the dispersion formula goes over into the classical formula for the scattering of a free electron.

For $n_0 \neq n$ we obtain from (11) the well-known formula for the Raman scattering as was predicted by Smekal and Heisenberg. The existence of a scattered radiation with a frequency shifted by an amount corresponding to the energy difference between two quantum states was experimentally discovered by Landsberg and Mandelstamm‡‡ (in solids) and Raman and Krishnan|||| (in liquid solutions).

2. *Coherence.* In the classical theory the radiation scattered by an atom is *coherent* with the primary radiation. This has its origin in the fact that, except in the resonance case, the *phase* of the scattered wave is the same as the phase of the primary beam (the phase difference δ is given by § 5 eq. (9). δ is zero except in the case of resonance).

The same is true also in the quantum theory, though of course only if the scattered frequency is the same as the primary one, i.e. if the atom returns to its initial state. The application of the idea of coherence requires, however, some care. As we have seen in § 7

† H. A. Kramers and W. Heisenberg, *Zs. f. Phys.* **31** (1925), 681.

‡ If the matrix elements are real we can write the first term also in the form

$$2p_{0n_0 n_i} p_{n_i n_0}(E_0 - E_i)/[(E_0 - E_i)^2 - k_0^2],$$

which corresponds exactly to § 5 eq. (11).

‖ I. Waller, *Zs. f. Phys.* **51** (1928), 213. Compare also Dirac, *Quantum Mechanics*, 2nd ed. Oxford 1935, p. 247.

†† In § 5 Θ denotes the angle between $\mathbf{k}$ and the direction of polarization of $\mathbf{k}_0$. In § 5 eq. (4) the summation is taken over the directions of polarization of $\mathbf{k}$.

‡‡ G. Landsberg and L. Mandelstamm, *Naturw.* **16** (1928), 557 and 772.

|||| C. V. Raman and K. S. Krishnan, *Nature*, **121** (1928), 501. Theory: for molecules G. Placzek, *Handb. d. Radiologie* VI. 2, Leipzig 1934; for solids I. Tamm, *Zs. f. Phys.* **60** (1930), 345. For other details see Kohlrausch: *The Raman Effect* (Berlin 1931).

the phase of a quantized light wave ϕ is only determined if the number of light quanta is undetermined corresponding to the uncertainty relation

$$\Delta N \Delta \phi = 1. \tag{13}$$

In the case of scattering of a single light quantum, as considered in subsection 1, the phases of the two waves are entirely undetermined. If, on the other hand, we determine the phases, we know nothing about the number of light quanta in the two waves, and we can thus hardly speak of a scattering at all.

We can, however, check the phase relations if we consider a primary wave with a large enough number of quanta to allow the determination of both the phases and the number of quanta with a comparatively high accuracy. (This is, of course, the ordinary transition to the classical theory.)

But for the scattering of a single quantum also, we can give the idea of coherence a simple physical meaning, if we consider the scattering by *two atoms* A, B, situated, say, at a distance $\mathbf{R}$ apart. In the classical theory the scattered waves of the two atoms interfere with each other, giving a maximum or minimum intensity according to the difference of light path of the two scattered waves. For this classical result, only the phase *difference* of the two waves scattered by the two atoms is essential. In the quantum theory the latter can have a definite value, even if the total number of light quanta is determined. In this case, however, we do not know from which *atom* the light quantum is scattered.

The same (classical) intensity distribution follows also from the quantum theory. We consider two similar atoms at the positions $\mathbf{X}_A$, $\mathbf{X}_B$ and with quantum states n_i, m_i. The distance between A and B may be denoted by

$$\mathbf{X}_A - \mathbf{X}_B = \mathbf{R} \tag{14}$$

(see Fig. 9). The transition probability for the scattering of a primary quantum $\mathbf{k}_0$ giving a secondary quantum $\mathbf{k}$ can be calculated in the same way as the scattering by a single atom. The radiation interacts with either of the two atoms, the interaction function is therefore given by

$$H' = H'_A + H'_B. \tag{15}$$

Furthermore, we obtain another set of intermediate states, corresponding to an absorption of $\mathbf{k}_0$ or emission of $\mathbf{k}$ by the second atom B connected with a transition of the atom B from m_0 to m_i.

The matrix elements of H' (15) can then be written down immediately. Instead of (8) we obtain two similar terms

$$H' = \frac{e^2}{\mu}\frac{2\pi\hbar^2 c^2}{\sqrt{(k_0 k)}}\left\{e^{i(\mathbf{\kappa}-\mathbf{\kappa}_0,\,\mathbf{X}_A)}\left[\frac{1}{\mu}\sum_i\left(\frac{p_{0n_0}n_i p_{n_i n_0}}{E_0-E_i+k_0}+\ldots\right)+\cos\Theta\right]+\right.$$

$$\left.+e^{i(\mathbf{\kappa}-\mathbf{\kappa}_0,\,\mathbf{X}_B)}\left[\frac{1}{\mu}\sum_i\left(\frac{p_{0m_0}m_i p_{m_i m_0}}{E_0-E_i+k_0}+\ldots\right)+\cos\Theta\right]\right\}. \quad (16)$$

The two brackets [ ] are equal because the two atoms were assumed to be similar. (16) is therefore identical with the matrix element (8), the only difference being that the factor $\exp i(\mathbf{\kappa}-\mathbf{\kappa}_0, \mathbf{X})$ is replaced by the factor

$$e^{i(\mathbf{\kappa}-\mathbf{\kappa}_0,\,\mathbf{X}_A)}+e^{i(\mathbf{\kappa}-\mathbf{\kappa}_0,\,\mathbf{X}_B)} = e^{i(\mathbf{\kappa}-\mathbf{\kappa}_0,\,\mathbf{X}_B)}(1+e^{i(\mathbf{\kappa}-\mathbf{\kappa}_0,\,\mathbf{R})}). \quad (17)$$

For the transition probability (12) we obtain, therefore, a factor

$$|1+e^{i(\mathbf{\kappa}-\mathbf{\kappa}_0,\,\mathbf{R})}|^2 = 2[1+\cos(\mathbf{\kappa}-\mathbf{\kappa}_0,\mathbf{R})]. \quad (18)$$

This is exactly the result which is to be expected from the classical theory. The scalar product $(\mathbf{\kappa}-\mathbf{\kappa}_0,\mathbf{R})$ represents the difference of light path for the two scattered waves in units $1/\lambda$, i.e. the classical difference of the phases (Fig. 9). The probability for the light quantum to be scattered in the direction $\mathbf{\kappa}$ is four times the probability of scattering by a single atom if $(\mathbf{\kappa}-\mathbf{\kappa}_0,\mathbf{R})$ is an integral multiple of 2π. (18) vanishes if $(\mathbf{\kappa}-\mathbf{\kappa}_0,\mathbf{R})$ is an odd integral multiple of π. Thus, the two waves scattered by the atoms A and B can be considered as *coherent* in the same sense as in the classical theory.†

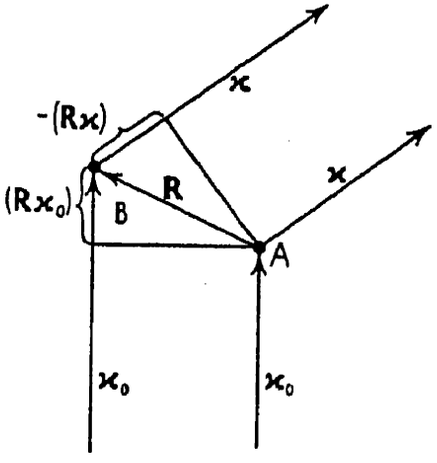

FIG. 9. Phase differences in the scattering of two atoms A, B.

3. *Scattering of X-rays.* Finally, we shall discuss qualitatively the behaviour of the scattering if the energy of the primary quantum $\mathbf{k}_0$ increases so that the wave-length becomes comparable with or smaller than the dimensions of the atom. In this case the exponential function $\exp i(\mathbf{\kappa}_0\mathbf{r})$ representing the light wave cannot be considered as constant in the integrals of the matrix elements (3) and (5).

The numerical value of the matrix elements *decreases* when these exponential functions vary appreciably inside the atom. According

† Compare also G. Wentzel, *Handb. d. Physik*, 2nd ed., xxiv (1).

to (12) the *scattered intensity decreases* in the same way. This is true for the coherent scattering as well as for the Raman scattering (at least if the atom is left in a discrete quantum state). Finally, if the wave-length λ_0 is small compared with the dimensions of the atom, the matrix elements and therefore the intensity of the scattered wave *vanish*. This will roughly be the case if

$$k_0 \gg \frac{\hbar c}{a} \sim 2 \times 137 \frac{I}{Z}, \tag{19}$$

where I represents the ionization energy and a the radius of the atom. For light elements (19) is satisfied in the region of hard X-rays.

On the other hand, as k_0 increases to values larger than I, another process becomes progressively more important. For $k_0 \gg I$ the electron can be left after the scattering in a state of the *continuous spectrum* with, say, a momentum $\mathbf{p}$ and an energy E. This is a certain kind of Raman effect. The frequency of the scattered radiation is then displaced relative to k_0, according to the formula

$$k = k_0 - (E - E_0). \tag{20}$$

Since the electron has a continuous energy spectrum, we obtain besides the ordinary undisplaced line (coherent scattering) another—displaced—line with a very broad intensity distribution. The total intensity will, however, at this stage be very small.

If now k_0 increases further, becoming large compared with I, the displaced line becomes increasingly sharper and more intense. This can be seen in the following way: We choose for the final state of the electron a state for which the momentum is determined by

$$\mathbf{p} \simeq \mathbf{k}_0 - \mathbf{k}. \tag{21}$$

Since then $\psi_n = \exp(i\mathbf{p}/\hbar c)$ in the integral (3), the factor $\exp i(\boldsymbol{\varkappa} - \boldsymbol{\varkappa}_0, \mathbf{r})$ is just compensated and (3) becomes *large*, however small the primary wave-length may be. For a given angle of scattering, k is determined entirely by (20) and (21). Thus we obtain an *intense and sharp displaced line*.

For very short wave-lengths, the process considered here becomes identical with the *scattering of a free electron* (Compton effect) which will be considered in detail in § **16**. (21) expresses simply the law of conservation of momentum (which is always satisfied in the interaction with free electrons), since for $k_0 \gg I$ the momentum of the electron in the bound state is relatively very small. The breadth of

the displaced (Compton) line is determined by the fluctuation of the momentum in the bound state.

The continuous transition from the coherent scattering of a bound electron to the Compton scattering of a free electron is shown qualitatively in Fig. 10. The decrease in the intensity of the undisplaced (coherent) line is proportional to the increase in the sharpness and intensity of the displaced line.†

The case (*c*) or (*d*) of Fig. 10 is realized approximately when X-rays of 50,000 volts energy are scattered in light elements (carbon, beryllium). According to measurements of Du Mond‡ the displaced

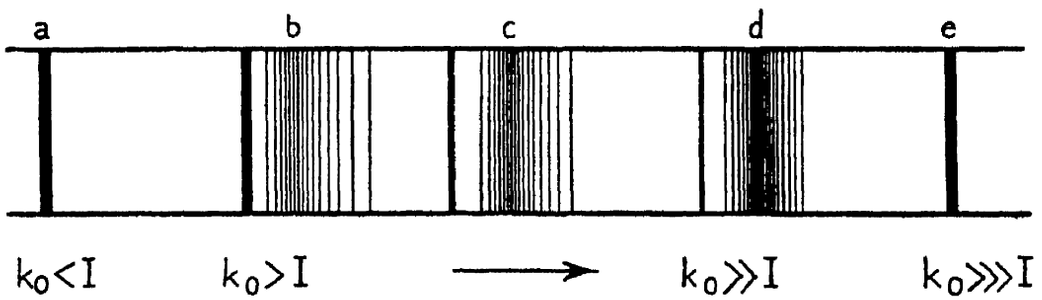

FIG. 10. Qualitative scheme of the coherent scattering and Compton scattering for increasing primary frequency k_0 (I = ionization energy of the atom) for a given angle. Case (*a*), coherent scattering only; case (*e*), Compton scattering of a free electron.

(Compton) line has a broad intensity distribution with a breadth of the order of magnitude of the displacement itself. The breadth has been shown to agree with that which one would expect from the momentum distribution of the electrons in the atom.

15. Resonance fluorescence

The theory of dispersion of § 14 breaks down if the frequency of the primary radiation k_0 approaches a resonance frequency of the atom $E_i - E_0$. In this case one of the denominators in the dispersion formula § 14 (12) vanishes and the intensity of the scattered radiation becomes infinite.

The reason for this break-down can be seen from the classical theory of dispersion by a harmonic oscillator, § 5 subsection 2. There the

† Report about the experimental results on the scattering of X-rays in *Handb. d. Physik*, xxiii. 2, article W. Bothe and F. Kirchner.

Calculations on the intensity of the coherent scattering: H. Hönl, *Zs. f. Phys.* **84** (1933), 1; *Ann. d. Phys.* **18** (1933), 625.

Incoherent scattering: W. Heisenberg, *Phys. Zs.* **32** (1931), 737; G. Wentzel, *Zs. f. Phys.* **43** (1927), 1, 779; F. Bloch, *Phys. Rev.* **46** (1934), 674.

‡ J. Du Mond, *Rev. of Modern Physics*, **5** (1933), 1; P. A. Ross and P. Kirkpatrick, *Phys. Rev.* **46** (1934), 668.

infinity in the neighbourhood of the resonance frequency ν_0 was avoided by taking into account the *damping force* due to the reaction of the emitted light on the atom. The procedure in the quantum theory is the same. Since this damping force is very small the intensity of the scattered radiation will in any case be very large compared with the ordinary scattering. This phenomenon, which is called resonance fluorescence, is treated in a way characteristic of the quantum mechanics and we shall therefore consider it here in some detail.

The radiative damping can be introduced in the dispersion formula in almost the same way as in the theory of line breadth, § **12**. We consider again the general equations for the probability amplitudes, § **9** subsection 3, and must solve them by taking into account the finite lifetimes of all states involved.

1. *General solution of the equations.* Since the resonance fluorescence will depend decidedly upon the intensity distribution of the primary radiation in the region of the natural line breadth, we shall for the present assume a general form for the primary intensity distribution $I_0(\nu)\,d\nu$ (energy per cm.^{2} and sec.) which will be specified later.

As we see from the dispersion formula, § **14** (12), in the case of resonance the only important intermediate state is that which has a vanishing denominator, and the scattering due to the quadratic term A^2 (second term in the dispersion formula) can be neglected. Denoting the ground state of the atom by n_0 (energy E_0) and the excited state in question by n (energy E) (we assume that neither state is degenerate), we can confine ourselves to those intermediate states where the atom is excited and one light quantum k_λ is absorbed. k_λ will *nearly* coincide with the resonance frequency of the atom, which we shall denote by

$$\nu_\lambda \sim (E-E_0)/\hbar \equiv \nu_0. \tag{1}$$

In the final state the atom will be again in the state E_0 and another light quantum $\mathbf{k}_\sigma$ will be emitted. For the present we do not know whether the frequency ν_σ is *exactly* identical with the absorbed frequency ν_λ, but, of course, ν_σ can differ from ν_λ only by an amount of the order of magnitude of the natural line breadth.

Denoting the probability amplitudes of the initial, intermediate, and final states by

$$b_0(t), \quad b_\lambda(t), \quad b_{\lambda\sigma}(t), \tag{2}$$

respectively, the differential equations for the b's, according to § 9 (32), are given by

$$-i\hbar\dot{b}_0(t) = \sum_\lambda H_{0,\lambda} b_\lambda(t) e^{i(\nu_0-\nu_\lambda)t} \tag{3a}$$

$$-i\hbar\dot{b}_\lambda(t) = H_{\lambda,0} b_0(t) e^{i(\nu_\lambda-\nu_0)t} + \sum_\sigma H_{\lambda,\lambda\sigma} b_{\lambda\sigma}(t) e^{i(\nu_\sigma-\nu_0)t} \tag{3b}$$

$$-i\hbar\dot{b}_{\lambda\sigma}(t) = H_{\lambda\sigma,\lambda} b_\lambda(t) e^{i(\nu_0-\nu_\sigma)t}. \tag{3c}$$

The notation of the matrix elements of H' corresponds to the notation of the probability amplitudes (2); the dashes in H' have been dropped. As initial conditions for the b's we assume

$$b_0(0) = 1, \qquad b_\lambda(0) = b_{\lambda\sigma}(0) = 0, \tag{4}$$

i.e. initially the atom is in the ground state and no light quantum is absorbed.

Following a procedure of Weisskopf† we try to solve the differential equations (3), putting

$$b_0(t) = e^{-\Gamma t/2} \tag{5a}$$

$$b_\lambda(t) = \beta_\lambda[e^{-\Gamma t/2} e^{i(\nu_\lambda-\nu_0)t} - e^{-\gamma t/2}]. \tag{5b}$$

Equations (5) satisfy the initial conditions (4). Clearly, Γ represents the *lifetime of the initial state.* The significance of γ will become clear later. γ could also depend upon λ, but we shall see that this is not the case. β_λ is a constant.

Inserting (5 b) into (3 c) we obtain

$$-i\hbar\dot{b}_{\lambda\sigma} = H_{\lambda\sigma,\lambda}\beta_\lambda[e^{-\Gamma t/2} e^{i(\nu_\lambda-\nu_\sigma)t} - e^{-\gamma t/2} e^{i(\nu_0-\nu_\sigma)t}]. \tag{6}$$

The solution of this equation, which satisfies the initial condition (4), is given by

$$b_{\lambda\sigma}(t) = H_{\lambda\sigma,\lambda}\beta_\lambda\left[\frac{e^{-\Gamma t/2} e^{i(\nu_\lambda-\nu_\sigma)t} - 1}{k_\lambda - k_\sigma + i\hbar\Gamma/2} - \frac{e^{-\gamma t/2} e^{i(\nu_0-\nu_\sigma)t} - 1}{k_0 - k_\sigma + i\hbar\gamma/2}\right]. \tag{7}$$

In order to determine the constants γ and β_λ we insert (7) and (5) into (3 b). Taking into account the fact that $H_{\lambda,\lambda\sigma} = H^*_{\lambda\sigma,\lambda}$ and multiplying by $\exp i(\nu_0-\nu_\lambda)t$, we obtain

$$\beta_\lambda[(k_\lambda - k_0 + i\hbar\tfrac{1}{2}\Gamma)e^{-\Gamma t/2} - i\hbar\tfrac{1}{2}\gamma\, e^{-\gamma t/2} e^{i(\nu_0-\nu_\lambda)t}]$$

$$= H_{\lambda,0}\, e^{-\Gamma t/2} + \beta_\lambda \sum_\sigma |H_{\lambda,\lambda\sigma}|^2 \left[\frac{e^{-\Gamma t/2} - e^{i(\nu_\sigma-\nu_\lambda)t}}{k_\lambda - k_\sigma + i\hbar\Gamma/2} - \frac{e^{-\gamma t/2} e^{i(\nu_0-\nu_\lambda)t} - e^{i(\nu_\sigma-\nu_\lambda)t}}{k_0 - k_\sigma + i\hbar\gamma/2}\right]. \tag{8}$$

† V. Weisskopf, *Ann. d. Phys.* **9** (1931), 23. We consider here the scattering of light by an atom in the ground state. The scattering by an excited atom has also been investigated by Weisskopf, *Zs. f. Phys.* **85** (1933), 451.

The summation $\sum_\sigma$ on the right-hand side has to be carried out over all quanta $\mathbf{k}_\sigma$ which can possibly be emitted. It may be replaced by an integral by a method often used in previous chapters (see for instance § **12** eqq. (8)–(11). Since the γ's will be small the integral has a sharp maximum at $k_\sigma \sim k_0$ or k_λ, respectively. $H_{\lambda,\lambda\sigma}$ represents the matrix element for the emission of $\mathbf{k}_\sigma$ (it does not actually depend on the light quantum $\mathbf{k}_\lambda$ which has been absorbed before). $H_{\lambda,\lambda\sigma}$ can therefore be considered as a constant during the integration. Since $k_\sigma \sim k_\lambda \sim k_0$ we can insert in $|H_{\lambda,\lambda\sigma}|^2$ the value at the resonance frequency k_0. Denoting again by $\rho_{k_0}\,d\Omega$ the number of radiation oscillators per unit volume of frequency ν_0, etc., and by $\overline{|H(k_0)|^2}$ the integral of $\int |H_{\lambda,\lambda\sigma}|^2\,d\Omega$ over all directions of propagation and polarization of the emitted quantum, we obtain for the integral in (8)

$$\int |H|^2 \rho_{k_\sigma}\,dk_\sigma\,d\Omega\,\frac{e^{-\Gamma t/2}-e^{i(\nu_\sigma-\nu_\lambda)t}}{k_\lambda-k_\sigma+i\hbar\Gamma/2} = \overline{|H|^2}\rho_{k_0}\,i\pi e^{-\Gamma t/2}, \tag{9}$$

or, inserting (9) into (8),

$$\beta_\lambda[(k_\lambda-k_0+i\hbar\tfrac{1}{2}\Gamma)e^{-\Gamma t/2}-i\hbar\tfrac{1}{2}\gamma\, e^{-\gamma t/2}e^{i(\nu_0-\nu_\lambda)t}] = H_{\lambda,0}\,e^{-\Gamma t/2}+i\pi\beta_\lambda\rho_{k_0}\overline{|H(k_0)|^2}[e^{-\Gamma t/2}-e^{-\gamma t/2}e^{i(\nu_0-\nu_\lambda)t}]. \tag{10}$$

Since (10) is valid for all times we can equate the coefficients of equal time factors and obtain

$$\gamma = \frac{2\pi}{\hbar}\overline{|H(k_0)|^2}\rho_{k_0} = w_{nn_0} \tag{11}$$

$$\beta_\lambda = \frac{H_{\lambda,0}}{k_\lambda-k_0+i\hbar(\Gamma-\gamma)/2}. \tag{12}$$

According to § **11** eq. (5) γ represents the *total transition probability for the spontaneous emission of a light quantum k by the atomic transition* $n \to n_0$ or the natural breadth of the emission line. γ therefore does not depend upon the light quantum absorbed previously. In the expression (12) for β_λ we have made use of (11).

The probability that, after a time t, k_λ has been absorbed and k_σ emitted is given by $|b_{\lambda\sigma}(t)|^2$. The intensity distribution of the emitted and absorbed line will finally be given by the value of $|b_{\lambda\sigma}|^2$ for the time $t = \infty$ when the absorption and re-emission process is certainly finished. According to (7) and (12) we have†

$$|b_{\lambda\sigma}(\infty)|^2 = \frac{|H_{\lambda,0}|^2|H_{\lambda\sigma,\lambda}|^2}{[(k_\lambda-k_\sigma)^2+\hbar^2\Gamma^2/4][(k_0-k_\sigma)^2+\hbar^2\gamma^2/4]}. \tag{13}$$

† We neglect here the possibility that after some time the atom can absorb a second quantum k_λ. Since, in general, the probability of absorption is very small

Before discussing our result (13) we have to satisfy the differential equation (3a). Inserting (5a), (5b), and (12) into (3a) we obtain

$$\Gamma e^{-\Gamma t/2} = \frac{2}{i\hbar} \sum_{\lambda} |H_{\lambda,0}|^2 \frac{e^{-\Gamma t/2} - e^{-\gamma t/2} e^{i(\nu_0 - \nu_\lambda)t}}{k_\lambda - k_0 + i\hbar(\Gamma - \gamma)/2}. \tag{14}$$

The summation on the right-hand side has to be carried out over all quanta which can possibly be *absorbed*. This summation will depend, of course, on the primary intensity distribution $I_0(\nu)\,d\nu$. We shall confine ourselves to two important cases: (*a*) The primary intensity is constant in the region of the natural line breadth, i.e. we irradiate the atom with a *continuous spectrum*. (*b*) The primary radiation consists of a monochromatic line, which is *sharp* compared with the breadth γ of the natural line of the atom.

2. *Case* (*a*). *Continuous absorption*. If $I_0(\nu)$ is constant in the region of the natural breadth, the summation $\sum_\lambda$ in (14) can again be replaced by an integral. $H_{0,\lambda}$ represents the matrix element for the absorption of a quantum k_λ. According to § **10** it is proportional to the number of quanta n_λ in the radiation oscillator λ.

We denote the average value of $|H_{0,\lambda}|^2$ over all oscillators from which a quantum $\mathbf{k}$ can be absorbed and over all orientations of the atom by $d\Omega |H(k)|^2 \bar{n}_\nu$.†

Applying again the formula for integrals of the type (9) we obtain from (14)

$$\Gamma = \frac{2\pi}{\hbar} \rho_{k_0}\, d\Omega\, |H(k_0)|^2 \bar{n}_{\nu_0}. \tag{15}$$

n_{ν_0} can be expressed by the primary intensity (§ **11** eq. (16))

$$\rho_{k_0}\, d\Omega\, \bar{n}_{\nu_0} = \frac{I_0(\nu_0)}{\hbar k_0 c}. \tag{16}$$

Hence we have $$\Gamma = \frac{2\pi}{\hbar^2 k_0 c} |H(k_0)|^2 I_0(\nu_0) = w_{n_0 n}. \tag{17}$$

The expression (15) or (17) for Γ is identical with the formula (17) § **11** for the *total probability of absorption per unit time*. Since Γ was defined as the reciprocal of the lifetime of the initial state we

and much smaller than that of emission, we can choose a time at which the atom has absorbed and emitted with certainty and where the probability for a second absorption is still negligible.

† Note that the integral of $|H(k)|^2$ over all directions of propagation and polarization was denoted by $\int d\Omega |H(k)|^2 = \overline{|H(k)|^2}$. Apart from the factor $\bar{n}_\nu$, $|H(k)|^2$ is identical for emission and absorption.

see that the total probability for the resonance fluorescence is equal to the total probability of absorption. In general, i.e. if the primary intensity is not extremely high, Γ will be very small compared with the transition probability γ for the spontaneous emission. Γ represents the natural breadth of the ground state of the atom due to the probability of absorption. If, however, the primary intensity is small the ground state is practically sharp.

We shall discuss now the formula (13) for the probability of the emission of a quantum k_σ and the absorption of a quantum k_λ.

Since Γ is very small the first factor in the denominator shows that k_λ may differ hardly at all from k_σ, the difference being at most of the order $\hbar\Gamma$ which is the breadth of the ground state. The *energy* is therefore *conserved* in so far as this is allowed by the uncertainty relation.

If we integrate equation (13) over all quanta k_σ which can be emitted, we obtain the shape of the absorption line, and if we integrate over all k_λ, the shape of the emission line.

The probability for the emission of a quantum k_σ becomes according to (15)

$$\sum_\lambda |b_{\lambda\sigma}(\infty)|^2 = w(k_\sigma) = \frac{\overline{|H(k_0)|^2}}{(k_0-k_\sigma)^2+\hbar^2\gamma^2/4}. \tag{18}$$

This is the same formula as that deduced in § **12** for the *shape* of a line emitted *spontaneously*. Thus, if we irradiate an atom with a continuous radiation, we obtain the same emission line as if we excite the atom in any other way, say by collisions.

Formula (18) appears as a simple generalization of the dispersion formula § **14** (12) for the neighbourhood of a resonance frequency and, on the other hand, corresponds to the classical dispersion formula § **5** (12) for an oscillator, in the sense of the correspondence principle.

The probability that after a time $t = \infty$ a quantum k_λ has been *absorbed* is given by the summation of (13) over all k_σ. Since $\Gamma \ll \gamma$ we can consider the second factor in the denominator as nearly constant in the region in which the first factor has its maximum. We obtain, according to (15),

$$\sum_\sigma |b_{\lambda\sigma}(\infty)|^2 = w(k_\lambda) = \frac{\overline{|H(k_0)|^2}}{(k_0-k_\lambda)^2+\hbar^2\gamma^2/4}. \tag{19}$$

The total probability of absorption and emission for $t = \infty$, $\sum_{\sigma,\lambda} |b_{\lambda\sigma}(\infty)|^2$, is of course equal to one.

(19) is again identical with the formula deduced in § 12 for the *shape* of the *absorption line* (and also identical with the shape (18) of the emission line).

We can therefore conclude the following: In the case where we irradiate with a continuous spectrum, the resonance fluorescence behaves with regard to the shape of the line, which is finally absorbed and emitted exactly as if two *independent processes, an absorption and a subsequent emission*, took place. For a single process, we have, however, to keep in mind that *energy is always conserved* and that therefore the atom 'remembers' before the emission which quantum it has absorbed. This is expressed by the fact that the formula (13) for $|b_{\lambda\sigma}(\infty)|^2$ is not identical with the product of the probabilities of

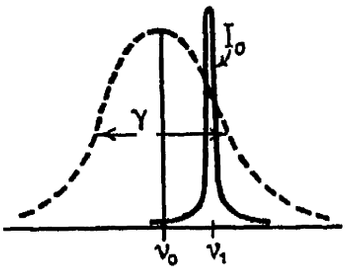

FIG. 11. Resonance fluorescence through excitation with a line $I_0(\nu)$ which is sharp compared with the natural breadth γ. The re-emitted line has the same shape $I_0(\nu)$.

emission and absorption since the first factor of the denominator connects the frequencies of the two quanta k_λ and k_σ and not k_λ with k_0.†

The dependence of the emitted on the absorbed quantum becomes more significant in the case of *monochromatic absorption.*

3. *Case (b). Excitation by sharp line.* We assume now that the primary line is *sharp* compared with the natural emission line. Then $I_0(\nu)$ is different from zero only, say, at a frequency ν_1 (see Fig. 11). We denote the total primary intensity $\int I_0(\nu)\, d\nu$ by $\bar{I}_0$.

Equations (3a) or (14) cannot be satisfied exactly in this case. The solution (5) and (7) is only approximate, being valid for $\Gamma \ll \gamma$. If we neglect $\exp(-\gamma t/2)$ in the numerator and Γ in the denominator of (14), the summation $\sum_\lambda$ reduces to a single term for $k_\lambda = k_1$. Since, according to (16), in our case

$$\int |H(k_\lambda)|^2 \rho_{k_\lambda} \bar{n}_{\nu_\lambda}\, d\Omega dk_\lambda = \frac{\overline{|H(k_1)|^2}\, \bar{I}_0}{k_1 c},$$

† If emission and absorption were independent, the first factor of the denominator would contain $(k_\lambda - k_0)^2$ instead of $(k_\lambda - k_\sigma)^2$.

we obtain from (14)

$$\Gamma = \frac{2}{i\hbar k_1 c} \frac{\overline{|H(k_1)|^2}\, \bar{I}_0}{(k_1 - k_0 - i\hbar\gamma/2)}. \tag{20}$$

Since Γ is now complex, the lifetime of the initial state is given by the *real part* of (20)

$$\mathrm{R}(\Gamma) = \frac{\gamma}{k_1 c} \frac{\overline{|H(k_1)|^2}\, \bar{I}_0}{(k_1 - k_0)^2 + \hbar^2\gamma^2/4}. \tag{21}$$

Equation (21) gives the *total probability for the resonance fluorescence* per unit time. It decreases with the distance of the primary frequency k_1 from the resonance frequency k_0 according to a dispersion formula with the half-value breadth γ (= breadth of emission line).

The intensity of the re-emitted line is again given by integrating (13) over all quanta which can possibly be absorbed. Since $\Gamma \ll \gamma$ the first factor in the denominator shows again that k_σ is practically equal to k_λ, or since only quanta of the frequency ν_1 can be absorbed, $k_\sigma = k_1$. Hence it follows that the emitted line must have the *same breadth as the primary line* and must therefore be much *sharper than the natural line.* We obtain the exact shape from (13) by integrating over all k_λ. Since Γ vanishes for small primary intensities we can assume that $I_0(\nu)$ is constant in a region of the breadth Γ (although, of course, I_0 is sharper than γ). The integration yields then, according to (16),

$$w(k_\sigma) = \sum_\lambda |b_{\lambda\sigma}(\infty)|^2 = \frac{2\pi}{\hbar^2 \Gamma k_0 c} \frac{(\overline{|H(k_0)|^2})^2\, I_0(k_\sigma)}{[(k_0 - k_\sigma)^2 + \hbar^2\gamma^2/4]}. \tag{22}$$

The intensity distribution $w(k_\sigma)$ is determined by two factors: Firstly, $w(k_\sigma)$ is proportional to $I_0(k_\sigma)$. This shows that *the emitted line has exactly the same shape* as the *primary line* and is therefore much sharper than the natural line (see Fig. 11). Secondly, the denominator of (22) is practically constant in the region where I_0 is different from zero. This factor determines the total intensity. The total intensity decreases with increasing distance from the maximum k_0 in the same proportion as the intensity of the natural line.

4. *Coherence and similar questions.* In the case (*b*) of a monochromatic excitation of the atom the emitted line has a quite different shape from that which the atom emits spontaneously. Emission and absorption cannot therefore be regarded as two subsequent *independent* processes, because the atom would not then 'remember' which light quantum it has absorbed before and would emit the

natural line. In this case, the resonance fluorescence has to be considered as a *single-quantum process* like the ordinary dispersion.

This is also confirmed by the following fact: By a method similar to that applied in § 14 subsection 2 it can easily be shown that the *re-emitted radiation is coherent with the primary radiation.*† This again would not be the case if absorption and emission were independent.

On the other hand, it may be asked, in which *state*—ground state or excited state—*the atom* is during the process of resonance fluorescence. This question can only be decided by a *measurement* of the energy of the atom. We shall see that by such a measurement all phase relations are destroyed entirely.

The energy of the atom can, for instance, be measured by inelastic collisions with electrons. To decide unambiguously whether the atom is excited or not the measurement must be carried out in a time which is shorter than the lifetime $1/\gamma$ of the excited state (otherwise, the atom would jump down spontaneously during the measurement). Therefore we must have at least one collision in the time $1/\gamma$. At the instant of time at which the collision takes place the *coherence* of the wave is certainly *interrupted.* (Shifting of phases by an uncertain amount.) But if we interrupt a light wave $1/\gamma$ times per second the line is no longer monochromatic but has a breadth $\hbar\gamma$.‡ This is just the breadth of the natural line. As a result of the measurement of the energy we see, therefore, that the emitted line becomes broader and has at least the breadth of the natural line. The process of resonance fluorescence behaves now just as if the atom emitted the light quantum spontaneously after having been excited by absorption of a primary quantum.

Thus we have found the following:

Resonance fluorescence represents a single coherent quantum process if the atom is undisturbed. For excitation by a sharp line the emitted line has then the same shape as the primary one. The energy of the atom is undetermined. As soon as the quantum state of the atom is determined the process behaves as an independent absorption and emission of a light quantum. The emitted line has then the natural shape.||

† The phase of the scattered radiation is, however, shifted against the phase of the primary radiation, as is the case in the classical theory (§ 5).

‡ Cf. Lorentz's classical theory of broadening of spectral lines by collisions (§ 4 subsection 5).

|| Even if the atom is excited by a sharp monochromatic beam of electrons with an energy E differing from k_0 by an amount of the order of γ, the excitation of the atom and the emission of a light quantum are 'coherent' and have to be considered as a

16. Scattering by free electrons

The dispersion formula (11) deduced in § **14** is valid if the primary frequency is of the order of magnitude of atomic frequencies. If, on the other hand, the energy of the primary quantum is large compared with the ionization energy, the electron in the atom can be considered as *free*. Scattering by free electrons is of fundamental importance in all phenomena connected with the absorption of γ-rays, cosmic radiation, etc., and we shall therefore investigate it here in some detail. Since we are interested especially in the high-energy region we must, of course, carry through all calculations relativistically.

1. *The Compton formula.* The process discussed here is the following: A primary light quantum $\mathbf{k}_0$ collides with a free electron which we can assume to be initially at rest:

$$\mathbf{p}_0 = 0, \qquad E_0 = \mu \qquad (\mu = mc^2). \tag{1}$$

The general case $\mathbf{p}_0 \neq 0$ can be obtained from the special case (1) by a Lorentz transformation. In the final state the light quantum has been scattered, so that we have instead of $\mathbf{k}_0$ a quantum $\mathbf{k}$. Since according to § **10** subsection **3** the *momentum* is *conserved* in the interaction of light with free electrons, the electron gets a kick and has therefore in the final state a momentum $\mathbf{p}$ (energy E)

$$\mathbf{p} = \mathbf{k}_0 - \mathbf{k}. \tag{2}$$

The conservation of energy states that

$$E + k = k_0 + \mu. \tag{3}$$

According to (2) and (3) the frequency of the scattered quantum cannot be the same as that of the primary quantum. Using the relativistic relation between momentum and energy $p^2 = E^2 - \mu^2$ and denoting the *angle* between $\mathbf{k}_0$ and $\mathbf{k}$ by θ, we obtain from (2) and (3)

$$k = \frac{k_0 \mu}{\mu + k_0(1 - \cos\theta)}, \tag{4}$$

which is the well-known formula for the frequency shift of the scattered radiation. It shows that in the non-relativistic case, $k_0 \ll \mu$, the scattered and primary frequencies are the same. In the relativistic case the frequency shift increases with the angle of scattering θ. In the 'extreme relativistic' case, i.e. if the primary quantum k_0 is large compared with the rest energy of the electron ($k_0 \gg \mu$), we can

single-quantum process. As can be seen immediately from the conservation of energy, the shape of the line emitted is not then identical with the natural one. Cf. W. Heitler, *Zs. f. Phys.* **82** (1933), 146.

distinguish between two regions of θ. For very small angles k is again nearly equal to k_0:

$$k \sim k_0 \quad \text{if} \quad k_0(1-\cos\theta) \ll \mu. \qquad \text{E.R.} \quad (5)$$

For large angles, i.e. for $(1-\cos\theta)k_0 \gg \mu$, we have

$$k = \frac{\mu}{1-\cos\theta}. \qquad \text{E.R.} \quad (6)$$

In this case the scattered quantum is always only of the order μ whatever the primary frequency. The wave-length is of the order

$$\lambda = \frac{\hbar c}{k} \sim \frac{\hbar}{mc} \equiv \lambda_0. \qquad (7)$$

λ_0 is the universal Compton wave-length. The region (5), in which k is appreciably larger than μ, becomes smaller as the primary frequency k_0 increases.

2. *Intermediate states, transition probability.* To compute the transition probability from the initial state A ($\mathbf{k}_0, \mathbf{p}_0 = 0$) to the final state F ($\mathbf{k}, \mathbf{p}$), we must remember that our process is a two-quanta process which can happen only by passing through an intermediate state which can differ by one quantum only from the initial and the final states. Since for these intermediate states the momentum is conserved (but not the energy), the following two intermediate states are the only ones possible:

I. $\mathbf{k}_0$ is first absorbed. No light quantum is present. The electron has a momentum

$$\mathbf{p}' = \mathbf{k}_0. \qquad (8\,a)$$

$\mathbf{k}$ is emitted in the transition to the final state.

II. $\mathbf{k}$ is first emitted. Both quanta $\mathbf{k}_0$ and $\mathbf{k}$ are present. The electron has a momentum

$$\mathbf{p}'' = -\mathbf{k}. \qquad (8\,b)$$

$\mathbf{k}_0$ is absorbed in the transition to the final state.

The state of an electron moving with relativistic velocity is not determined completely by its momentum $\mathbf{p}$. In § 9 we have seen that for a free electron with a given momentum $\mathbf{p}$ altogether *four states* exist, corresponding to the fact that the electron may have either of two *spin directions* and also a *positive* or *negative energy*

$$E = \pm\sqrt{(p^2+\mu^2)}. \qquad (9)$$

The physical significance of the negative energy states will be discussed in Chapter IV, but there is no doubt that here, in our

problem, they must be taken into account as *intermediate states*. Each of the two intermediate states I and II is therefore actually *fourfold*, because by (8 a) and (8 b) only the momentum of the electron is determined. On the other hand, in the initial and final states the electron has of course a positive energy, and we shall assume for the present also a given spin direction.

The matrix element which determines the transition probability is then given by

$$H = \sum \left(\frac{H_{AI} H_{IF}}{E_A - E_I} + \frac{H_{AII} H_{IIF}}{E_A - E_{II}} \right), \tag{10}$$

where $\sum$ denotes the summation over all four intermediate states, i.e. over both spin directions and both signs of the energy. $E_A, E_I \ldots$ represent the total energies in the initial and the intermediate states. The energy differences occurring in the denominator of (10) are, according to (2), (3), (8), (9), given by

$$\begin{aligned} E_A - E_I &= \mu + k_0 - E' \\ E_A - E_{II} &= \mu + k_0 - (E'' + k_0 + k) = \mu - E'' - k, \end{aligned} \tag{11}$$

where E' represents the energy of the electron in the state I,

$$E' = \pm\sqrt{(p'^2 + \mu^2)}.$$

If we denote the Dirac amplitudes of the electron with the momenta $\mathbf{p}_0, \mathbf{p}, \mathbf{p}', \mathbf{p}''$ by u_0, u, u', u''† and the components of the matrix vector $\boldsymbol{\alpha}$ in the direction of the polarization of the two light quanta $\mathbf{k}_0$ and $\mathbf{k}$ simply by α_0 and α, respectively, the matrix elements for the transitions $A \to$ I, etc., are given by § 10 eq. (16):

$$\begin{aligned} H_{AI} &= -e\sqrt{\left(\frac{2\pi\hbar^2 c^2}{k_0}\right)}(u_0^* \alpha_0 u'), \qquad H_{IF} = -e\sqrt{\left(\frac{2\pi\hbar^2 c^2}{k}\right)}(u'^* \alpha u), \\ H_{AII} &= -e\sqrt{\left(\frac{2\pi\hbar^2 c^2}{k}\right)}(u_0^* \alpha u''), \qquad H_{IIF} = -e\sqrt{\left(\frac{2\pi\hbar^2 c^2}{k_0}\right)}(u''^* \alpha_0 u). \end{aligned} \tag{12}$$

The transition probability per unit time for our scattering process is, according to § 9 eq. (42), given by

$$w = \frac{2\pi}{\hbar} |H|^2 \rho_F, \tag{13}$$

where ρ_F denotes the number of final states per energy interval dE_F. The evaluation of ρ_F requires some care. By the conservation of momentum the final state is determined completely by the

† For simplicity we omit in the u's the indices denoting the spin direction and sign of energy.

frequency of the scattered quantum $\mathbf{k}$ and the angle of scattering. Therefore we have

$$\rho_F dE_F = \rho_k dk, \tag{14}$$

where ρ_k denotes the number of states for the scattered quantum per energy interval dk. It would be incorrect, however, to equate the energy intervals dk and dE_F. Since the final energy is given as a function of k and θ by

$$E_F = k+\sqrt{(p^2+\mu^2)} = k+(k_0^2+k^2-2k_0 k\cos\theta+\mu^2)^{\frac{1}{2}}, \tag{15}$$

we obtain

$$\left(\frac{\partial k}{\partial E_F}\right)_\theta = \frac{Ek}{\mu k_0}, \tag{16}$$

and hence

$$\rho_F = \rho_k\left(\frac{\partial k}{\partial E_F}\right)_\theta = \frac{d\Omega\, k^2}{(2\pi\hbar c)^3}\,\frac{Ek}{\mu k_0}. \tag{17}$$

$d\Omega$ represents the element of solid angle for the scattered quantum. Collecting our formulae (10), (11), (12), (13), (17) and dividing by the velocity of light, we obtain a quantity of the dimension cm.^{2}, the *differential cross-section for the scattering process*,

$$d\phi = e^4\frac{Ek^2}{\mu k_0^2}d\Omega\left[\sum\left(\frac{(u_0^*\alpha_0 u')(u'^*\alpha u)}{\mu+k_0-E'}+\frac{(u_0^*\alpha u'')(u''^*\alpha_0 u)}{\mu-k-E''}\right)\right]^2. \tag{18}$$

(18) is valid for a given polarization of both light quanta and a given spin direction of the electron in the initial and final states. The summation $\sum$ is over all spin directions and both signs of the energy for the intermediate states.

3. *Deduction of the Klein-Nishina formula.* Our next task is the evaluation of the matrix elements occurring in (18). The summation $\sum$ can easily be carried out if we make use of the general formula § 9 (25). This formula cannot, however, be applied directly to (18) since the denominators of (18) depend upon the sign of the energy E'. We therefore multiply the numerator and denominator of the first term of (18) by $\mu+k_0+E'$. The denominator does not then depend on the sign of E'. For the numerator, we make use of the wave equation

$$E'u' = [(\boldsymbol{\alpha}\mathbf{p}')+\beta\mu]u' \equiv H'u', \tag{19}$$

where H' represents an operator involving *linearly* the matrices α and β. $\mathbf{p}'$ is of course constant for the summation $\sum$. Thus we obtain according to (19)†

$$\sum(\mu+k_0+E')(u_0^*\alpha_0 u')(u'\alpha u) = (\mu+k_0)\sum(u_0^*\alpha_0 u')(u'^*\alpha u)+ \\ +\sum(u_0^*\alpha_0 H'u')(u'^*\alpha u). \tag{20}$$

† Compare for this method, H. Casimir, *Helv. Phys. Acta*, **6** (1933), 287.

Applying now our general formula § 9 (25)

$$\sum^{p'}(u_0^* O u')(u'^* Q u) = (u_0^* O Q u),$$

and taking into account the fact that $E'^2 = p'^2+\mu^2 = k_0^2+\mu^2$ we obtain, according to (20), for the first term of (18)

$$\sum^{p'}\frac{(u_0^*\alpha_0 u')(u'^*\alpha u)}{\mu+k_0-E'} = \frac{(\mu+k_0)(u_0^*\alpha_0\alpha u)+(u_0^*\alpha_0 H'\alpha u)}{2\mu k_0}. \tag{21}$$

In the same way the second term of (18) can be evaluated. If we introduce the abbreviations

$$K' = \mu+k_0+H' = \mu(1+\beta)+k_0+(\boldsymbol{\alpha}\mathbf{k}_0) \tag{22a}$$

$$K'' = \mu-k+H'' = \mu(1+\beta)-k-(\boldsymbol{\alpha}\mathbf{k}) \tag{22b}$$

(where we have replaced $\mathbf{p}'$ and $\mathbf{p}''$ by (8a) and (8b)), we obtain for the summation $\sum$ in (18)

$$\sum = \frac{1}{2\mu}\left[\frac{(u_0^*\alpha_0 K'\alpha u)}{k_0} - \frac{(u_0^*\alpha K''\alpha_0 u)}{k}\right]. \tag{23}$$

The differential cross-section (18) is proportional to the square of (23). (23) depends on the spin directions of the electron in the initial and final states. We are not, however, interested in the probability of finding the electron with a certain spin after the scattering process. We shall therefore sum $d\phi$ over all spin directions of the electron after the scattering process and shall average over the spin directions in the initial state.

We denote the summation over the spin directions only by S (or S_0 for the initial state), in contrast to the summation $\sum$ which is extended over both signs of energy too. Thus we must form

$$\tfrac{1}{2}S_0\,S\,|\text{expression (23)}|^2. \tag{24}$$

The summation S over the spin directions can be reduced to a summation $\sum^p$ over all four states having the same momentum p. According to the wave equation (19) we can replace u by

$$u = \frac{H+E}{2E}u. \tag{25}$$

(25) is correct for positive and negative E. For positive E we can also write

$$u = \frac{H+|E|}{2|E|}u. \tag{25'}$$

If now $\tilde{u}$ represents a wave function belonging to a negative energy state, the operator $H+|E|$ operating on $\tilde{u}$ gives zero:

$$(H+|E|)\tilde{u} = 0.$$

Therefore, if in a summation S we replace one wave function u by the expression (25), we can sum over all four states, including the states of negative energy, since the contributions of the latter vanish. Taking now the square of (23) we obtain expressions of the form

$$\mathsf{S}(u_0^* Au)(u^* Bu_0).$$

For these we can write according to (25) and § 9 (25)

$$\mathsf{S}(u_0^* Au)(u^* Bu_0) = \frac{1}{2|E|}\sum\{u_0^* A(H+|E|)u\}(u^* Bu_0)$$
$$= \frac{1}{2E}\{u_0^* A(H+E)Bu_0\}, \quad (26)$$

where we have again replaced $|E|$ by E (since E is positive). Applying the same method to the summation S_0 we obtain

$$\tfrac{1}{2}\mathsf{S}\mathsf{S}_0(u_0^* Au)(u^* Bu_0) = \frac{1}{8E_0E}\sum\nolimits_0\{u_0^* A(H+E)B(H_0+E_0)u_0\}$$
$$= \frac{1}{8E_0E}\,\mathrm{Sp}\,A(H+E)B(H_0+E_0). \quad (27)$$

'Sp' denotes (§ 9 eq. (26)) the '*spur*' (diagonal sum) of the operator $A(H+E)B(H_0+E_0)$.

If we take now the square of (23), the first term becomes, according to (27) and § 9 (24),

$$\tfrac{1}{2}\mathsf{S}_0\mathsf{S}\frac{1}{4\mu^2k_0^2}|(u_0^*\alpha_0K'\alpha u)|^2 = \frac{1}{8\mu^2k_0^2}\mathsf{S}_0\mathsf{S}(u_0^*\alpha_0K'\alpha u)(u^*\alpha K'\alpha_0u_0)$$
$$= \frac{1}{32\mu^2k_0^2E_0E}\,\mathrm{Sp}\,\alpha_0K'\alpha(H+E)\alpha K'\alpha_0(H_0+E_0). \quad (28)$$

The other terms arising from the square of (23) give similar expressions. Remembering that the electron was initially at rest $\mathbf{p}_0 = 0$, $E_0 = \mu$ and therefore that

$$H_0+E_0 = \mu(1+\beta),$$

we obtain finally for the differential cross-section from (18), (23), (28)

$$d\phi = \left(\frac{e^2}{\mu}\right)^2\left(\frac{k}{k_0}\right)^2 d\Omega\,\frac{1}{8\mu}\left[\frac{1}{4k_0^2}\,\mathrm{Sp}\,P+\frac{1}{4k^2}\,\mathrm{Sp}\,Q-\frac{2}{4k_0k}\,\mathrm{Sp}\,R\right], \quad (29)$$

where

$$P = \alpha_0K'\alpha(H+E)\alpha K'\alpha_0(1+\beta),$$
$$Q = \alpha K''\alpha_0(H+E)\alpha_0K''\alpha(1+\beta), \quad (30)$$
$$R = \alpha_0K'\alpha(H+E)\alpha_0K''\alpha(1+\beta).$$

As a last step in the computation of $d\phi$ we have to evaluate the spur

of (30). For this purpose the necessary formulae have been deduced in § 9 subsection 2 eq. (27).

We consider the first term Sp P. Since β anti-commutes with each α we have

$$\alpha_0(1+\beta) = (1-\beta)\alpha_0 \quad \text{(for each } \alpha\text{)}, \tag{31}$$

and since the factors in the spur can be permuted cyclically, we obtain, since $\alpha_0^2 = 1$,

$$\text{Sp } P = \text{Sp } K'\alpha(H+E)\alpha K'(1-\beta). \tag{32}$$

Furthermore, we have

$$(1+\beta)(1-\beta) = 1-\beta^2 = 0. \tag{33}$$

The first term $\mu(1+\beta)$ of K' (22a) gives therefore no contribution to (32):

$$\text{Sp } P = \text{Sp}[k_0+(\boldsymbol{\alpha}\mathbf{k}_0)]\alpha(H+E)\alpha[k_0+(\boldsymbol{\alpha}\mathbf{k}_0)](1-\beta). \tag{34}$$

For $H+E$ we insert

$$H+E = E+(\boldsymbol{\alpha}\mathbf{p})+\beta\mu \tag{35}$$

and make use of the fact that the spur of an operator containing an odd number of factors β vanishes. We obtain then from (34), making use of (31),

$$\text{Sp } P = \text{Sp } \alpha[E+(\boldsymbol{\alpha}\mathbf{p})]\alpha[k_0+(\boldsymbol{\alpha}\mathbf{k}_0)]^2+\mu \text{ Sp}[k_0+(\boldsymbol{\alpha}\mathbf{k}_0)][k_0-(\boldsymbol{\alpha}\mathbf{k}_0)].$$

The second term vanishes because the commutation relations of the α's (§ 9 (16)) give $(\boldsymbol{\alpha}\mathbf{k}_0)^2 = k_0^2$. Taking into account the fact that only operators with an even number of factors α have a spur different from zero, we obtain finally, according to § 9 (27 b),

$$\tfrac{1}{4} \text{Sp } P = 2Ek_0^2+\tfrac{1}{2}k_0 \text{ Sp } \alpha(\boldsymbol{\alpha}\mathbf{p})\alpha(\boldsymbol{\alpha}\mathbf{k}_0). \tag{36}$$

We need not evaluate the second term of (36) because it will cancel with a corresponding term arising from Sp R.

In the same way we obtain

$$\tfrac{1}{4} \text{Sp } Q = 2Ek^2+\tfrac{1}{2}k \text{ Sp } \alpha_0(\boldsymbol{\alpha}\mathbf{p})\alpha_0(\boldsymbol{\alpha}\mathbf{k}). \tag{37}$$

Sp R can be evaluated similarly. According to (31) and (33) the terms $\mu(1+\beta)$ of K' and K'' (22) give again no contribution. Hence we obtain, according to (35),

$$\begin{aligned}\text{Sp } R &= -\text{ Sp } \alpha\alpha_0[k_0+(\boldsymbol{\alpha}\mathbf{k}_0)]\alpha(H+E)\alpha_0[k+(\boldsymbol{\alpha}\mathbf{k})](1-\beta)\\ &= -\text{ Sp } \alpha\alpha_0[k_0+(\boldsymbol{\alpha}\mathbf{k}_0)]\alpha[E+(\boldsymbol{\alpha}\mathbf{p})]\alpha_0[k+(\boldsymbol{\alpha}\mathbf{k})]-\\ &\qquad -\mu \text{ Sp } \alpha\alpha_0[k_0+(\boldsymbol{\alpha}\mathbf{k}_0)]\alpha\alpha_0[k-(\boldsymbol{\alpha}\mathbf{k})].\end{aligned} \tag{38}$$

In (38) we make use of the fact that only terms with an even number of factors α give any contribution. Since, furthermore, the

direction of $\mathbf{k}$ is perpendicular to the direction of polarization, α and $(\boldsymbol{\alpha}\mathbf{k})$ anti-commute:

$$\alpha(\boldsymbol{\alpha}\mathbf{k}) = -(\boldsymbol{\alpha}\mathbf{k})\alpha. \tag{39}$$

We then obtain (permuting the factors cyclically in some of the terms)

$$\begin{aligned} \operatorname{Sp} R = & -(\mu+E)k_0 k \operatorname{Sp} \alpha\alpha_0\alpha\alpha_0 \\ & +(\mu-E) \operatorname{Sp} \alpha_0\alpha\alpha_0\alpha(\boldsymbol{\alpha}\mathbf{k})(\boldsymbol{\alpha}\mathbf{k}_0) \\ & -k_0 \operatorname{Sp} \alpha\alpha_0\alpha(\boldsymbol{\alpha}\mathbf{p})\alpha_0(\boldsymbol{\alpha}\mathbf{k}) \\ & -k \operatorname{Sp} \alpha_0\alpha\alpha_0(\boldsymbol{\alpha}\mathbf{k}_0)\alpha(\boldsymbol{\alpha}\mathbf{p}). \end{aligned} \tag{40}$$

In (40) the product $\alpha\alpha_0$ occurs. If the two directions of polarization are perpendicular to each other, α and α_0 anti-commute. This is in general not the case, but we can always represent α as a sum of two components, one of which is orthogonal to α_0 and the other has the same direction as α_0. The first component anti-commutes with α_0. The second component is equal to $\alpha_0 \cos\Theta$, where Θ represents the angle between the two directions of polarization, and commutes therefore with α_0 Thus we can write

$$\alpha\alpha_0 = -\alpha_0\alpha + 2\cos\Theta. \tag{41}$$

Accordingly we have

$$\tfrac{1}{4} \operatorname{Sp} \alpha_0\alpha = \cos\Theta. \tag{42}$$

We shall now change the order of one product $\alpha_0\alpha$ in each term of (40). Using the formula

$$\tfrac{1}{4} \operatorname{Sp} (\boldsymbol{\alpha}\mathbf{k})(\boldsymbol{\alpha}\mathbf{k}_0) = (\mathbf{k}_0\mathbf{k}), \tag{43}$$

which can immediately be deduced from § 9 eq. (27 b), we obtain

$$\begin{aligned} \tfrac{1}{4}\operatorname{Sp} R = & (\mu+E)k_0 k + (E-\mu)(\mathbf{k}_0\mathbf{k}) + \\ & + \tfrac{1}{4}k_0 \operatorname{Sp} \alpha_0(\boldsymbol{\alpha}\mathbf{p})\alpha_0(\boldsymbol{\alpha}\mathbf{k}) + \tfrac{1}{4}k \operatorname{Sp} \alpha(\boldsymbol{\alpha}\mathbf{p})\alpha(\boldsymbol{\alpha}\mathbf{k}_0) - \\ & - 2\cos\Theta\{k_0 k(E+\mu)\cos\Theta + (E-\mu)\tfrac{1}{4} \operatorname{Sp} \alpha_0(\boldsymbol{\alpha}\mathbf{k}_0)(\boldsymbol{\alpha}\mathbf{k})\alpha + \\ & + \tfrac{1}{4} \operatorname{Sp} \alpha(\boldsymbol{\alpha}\mathbf{p})\alpha_0[(\boldsymbol{\alpha}\mathbf{k})k_0 + (\boldsymbol{\alpha}\mathbf{k}_0)k]\}. \end{aligned} \tag{44}$$

In the bracket { } we insert for $\mathbf{p}$ the value $\mathbf{k}_0 - \mathbf{k}$ and make use of the conservation of energy (3). Taking into account (39), (42) and $(\boldsymbol{\alpha}\mathbf{k})^2 = k^2$, the term proportional to $\cos\Theta$ becomes simply

$$-2\cos\Theta\{\ \} = -4\cos^2\Theta k_0 k\mu. \tag{45}$$

Collecting now our formulae (36), (37), (44), (45) the total spur occurring in (29) becomes

$$\frac{1}{4k_0^2}\operatorname{Sp} P + \frac{1}{4k^2}\operatorname{Sp} Q - \frac{2}{4k_0 k}\operatorname{Sp} R = 2(k_0-k)\left(1-\frac{(\mathbf{k}_0\mathbf{k})}{k_0 k}\right) + 8\mu\cos^2\Theta, \tag{46}$$

and if we finally write for $(\mathbf{k}_0\mathbf{k})$ from (2) and (3)

$$k_0 k - (\mathbf{k}_0\mathbf{k}) = \mu(k_0 - k),$$

we obtain for the differential cross-section (29)

$$d\phi = \tfrac{1}{4} r_0^2 \, d\Omega \frac{k^2}{k_0^2}\left[\frac{k_0}{k} + \frac{k}{k_0} - 2 + 4\cos^2\Theta\right]. \tag{47}$$

(47) represents the well-known *Klein-Nishina formula.*† It gives, for all primary light quanta of a given frequency and polarization, the intensity of the scattered radiation at a given angle θ and with a given direction of polarization. Θ represents the angle between the directions of polarization of $\mathbf{k}_0$ and $\mathbf{k}$. (47) can be expressed by (4) as a function of k_0, θ, and Θ.

We have discussed in detail the computation of the matrix elements because the method used here will serve as a model for similar calculations in many other quantum processes.—The influence of damping on the Klein-Nishina formula will be discussed in § 25.

4. *Polarization, angular distribution.* We discuss first the angular distribution and polarization of the scattered radiation as given by formula (47). Since the frequency k varies rapidly with the angle of scattering, we must distinguish between the cross-section for the probability of scattering (given by (47)) and the intensity of the scattered radiation. The latter, at a distance R from the scattering electron, is given by

$$I\,d\Omega = I_0 \frac{d\phi}{R^2}\frac{k}{k_0}, \tag{48}$$

where I_0 represents the primary intensity (energy per cm.^{2} sec.).

It will be convenient to consider the scattered radiation as composed of two linearly polarized components $\perp$ and $\parallel$. Denoting the directions of polarization of $\mathbf{k}_0$ and $\mathbf{k}$ by $\mathbf{e}_0$ and $\mathbf{e}$ respectively, we can choose the following two directions for $\mathbf{e}$:

($\perp$) $\mathbf{e} \perp \mathbf{e}_0$, $\cos\Theta \equiv (\mathbf{e}_0\,\mathbf{e}) = 0$,

($\parallel$) $\mathbf{e}$ and $\mathbf{e}_0$ in the same plane (i.e. in the $(\mathbf{k}, \mathbf{e}_0)$ plane),

$$\cos^2\Theta = 1 - \sin^2\theta\cos^2\phi,$$

where ϕ represents the angle between the $(\mathbf{k}_0, \mathbf{k})$ plane and the $(\mathbf{k}_0, \mathbf{e}_0)$ plane and θ the angle of scattering $(\mathbf{k}_0, \mathbf{k})$. According to (47) the $\parallel$ component is always more intense than the $\perp$ component.

† O. Klein and Y. Nishina, *Zs. f. Phys.* **52** (1929), 853; Y. Nishina, ibid. **52** (1929), 869. The same formula has also been deduced by I. Tamm, ibid. **62** (1930), 545.

In the non-relativistic case (N.R.) we have $k_0 = k$ and

$$d\phi_\perp = 0, \quad d\phi_\parallel = r_0^2\, d\Omega(1-\sin^2\theta\cos^2\phi). \qquad \text{N.R.} \quad (49)$$

(49) is identical with the classical (Thomson) formula § 5 (4). This is to be expected because the condition $k_0 = \hbar\nu_0 \ll mc^2$ can also be interpreted as $\hbar \to 0$, which represents the transition to the classical theory. For polarized primary radiation the scattered radiation is completely polarized. If the primary radiation is unpolarized, we have to take the average over ϕ. The intensity of the scattered radiation is then given by

$$d\phi = \tfrac{1}{2} r_0^2\, d\Omega\,(1+\cos^2\theta). \qquad \text{N.R.} \quad (49')$$

In the other extreme case, where the energy of the primary quantum is large compared with mc^2 ($k_0 \gg \mu$, *extreme relativistic* case), we have according to subsection 1 to distinguish between small angles θ (5) and large angles (6). The differential cross-section becomes in these two cases (i.e. for $k_0 \sim k$ and $k_0 \gg k$)

$$d\phi_\perp = 0, \quad d\phi_\parallel \simeq r_0^2\, d\Omega\,(1-\sin^2\theta\cos^2\phi) \quad \text{(small angles)}, \quad \text{E.R.} \quad (50\,\text{a})$$

$$d\phi_\perp = d\phi_\parallel = \frac{r_0^2\, d\Omega}{4}\frac{k}{k_0} = \frac{r_0^2\, d\Omega\,\mu}{4k_0(1-\cos\theta)} \quad \text{(large angles)}. \quad \text{E.R.} \quad (50\,\text{b})$$

From (50) we see that for very small angles the intensity distribution is the same as the classical. For large angles, the scattered radiation is *unpolarized*, even if the primary radiation is polarized, and has roughly a uniform intensity distribution. The intensity decreases, however, with increasing energy of the primary quantum.

From (50 b) we can see already that, for $k_0 \gg \mu$, the total probability of scattering decreases $\sim \mu/k_0$. Both regions (θ small and large) give about the same contribution to the total cross-section, since the region in which (50 a) is valid is of the order $\theta^2 \sim \mu/k_0$ and $d\Omega = \theta\, d\theta$ (see subsection 5).

To obtain finally the total intensity scattered into an angle θ we have to take the sum $d\phi = d\phi_\perp + d\phi_\parallel$. If the primary radiation is unpolarized (average over ϕ), we obtain

$$d\phi = \frac{r_0^2\, d\Omega}{2}\frac{k^2}{k_0^2}\left(\frac{k_0}{k}+\frac{k}{k_0}-\sin^2\theta\right). \qquad (51)$$

Here k is a function of θ, given by (4). If we insert (4) in (51) the

scattered *intensity* becomes, according to (48),

$$R^2 \frac{I}{I_0} d\Omega = r_0^2 d\Omega \frac{1+\cos^2\theta}{2} \frac{1}{[1+\gamma(1-\cos\theta)]^3} \left\{1+\frac{\gamma^2(1-\cos\theta)^2}{(1+\cos^2\theta)[1+\gamma(1-\cos\theta)]}\right\}, \tag{52}$$

$$\gamma = k_0/\mu.$$

The intensity distribution (52) is plotted in Fig. 12 as a function of the angle of scattering θ for various ratios $\gamma = k_0/\mu$ (units r_0^2/R^2).

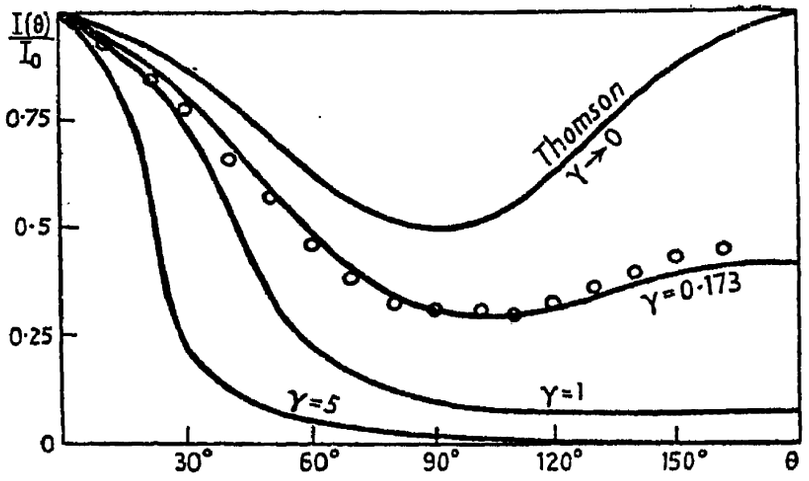

FIG. 12. Intensity distribution of the Compton scattering as a function of the angle of scattering for various primary frequencies $\gamma = k_0/\mu$. Measurements of Friedrich and Goldhaber for $\gamma = 0{\cdot}173$.

For small angles the scattered intensity has nearly the classical value, whereas for large angles the intensity is the smaller, the higher is the primary frequency. In the relativistic region the forward direction becomes more and more preponderant. Even for hard X-rays ($\gamma \sim 0{\cdot}2$) the relativistic deviation from the Thomson formula is rather large for large angles.

The hardest radiation for which the angular distribution has been measured is that of wave-length 0·14 Å.U. or $\gamma = 0{\cdot}173$. In Fig. 12 we have plotted the result of measurements by Friedrich and Goldhaber† of the intensity distribution for this radiation in carbon. In carbon for this radiation all the electrons can, of course, be considered as free. The total intensity has been divided by the total number of electrons. The agreement with the theoretical curve is exact (within the experimental error) and gives therefore a conclusive proof of the existence of relativistic deviations from the Thomson formula.

† W. Friedrich and G. Goldhaber, *Zs. f. Phys.* **44** (1927), 700. See also G. E. M. Jauncy and G. G. Harvey, *Phys. Rev.* **37** (1931), 698.

It should, however, be noticed that this agreement does not yet prove the Klein-Nishina formula, because other relativistic wave equations which were put forward before Dirac's give nearly the same curve for radiation with γ up to 0·2. The characteristic feature of the Klein-Nishina formula begins first at higher primary frequencies ($\gamma \sim 1$). We shall describe in subsection 5 a conclusive test of Dirac's wave equation from the measurements of the total scattering coefficient for hard γ-rays ($\gamma = 1$–5).

5. *Total scattering.* To obtain the total scattering we have to integrate over all angles. Here, again, we must distinguish between the total probability of scattering and the total scattered intensity. Since the absorption of the primary light beam is determined by the *number of scattered quanta* (each scattered quantum diminishes the primary intensity by an amount k_0) rather than by the scattered intensity, we integrate the differential cross-section (51) (inserting for k equation (4)) over the angles of scattering. An elementary integration yields for the total cross-section

$$\frac{\phi}{\phi_0} = \frac{3}{4}\left\{\frac{1+\gamma}{\gamma^3}\left[\frac{2\gamma(1+\gamma)}{1+2\gamma} - \log(1+2\gamma)\right] + \frac{1}{2\gamma}\log(1+2\gamma) - \frac{1+3\gamma}{(1+2\gamma)^2}\right\}, \tag{53}$$

$$\phi_0 = 8\pi r_0^2/3, \qquad \gamma = k_0/\mu.$$

The unit ϕ_0 used for ϕ is the classical cross-section for the Thomson scattering (§ 5 eq. (5)).

In the non-relativistic case $\gamma \ll 1$ we obtain again $\phi = \phi_0$. The first terms of an expansion of the right-hand side of (53) in powers of γ are

$$\phi = \phi_0\left(1 - 2\gamma + \frac{26}{5}\gamma^2 + \ldots\right). \qquad \text{N.R.} \quad (54)$$

In the extreme relativistic case, on the other hand, we obtain from (53)

$$\phi = \phi_0 \frac{3}{8}\frac{\mu}{k_0}\left(\log\frac{2k_0}{\mu} + \frac{1}{2}\right). \qquad \text{E.R.} \quad (55)$$

(55) agrees (apart from the logarithm) with the estimate of subsection 4, based on (50). *Thus for very high energies the number of scattered quanta decreases with the frequency of the primary radiation.* This is the reason why the penetrating power of γ-rays increases with increasing frequency—at least as long as no other absorption processes, such as the production of pairs (§ **20**), are important.

The cross-section (53) is plotted in Fig. 13 as a function of the primary energy on a logarithmic scale in order to cover a large energy region (the lower scale gives the wave-length of the primary radiation in X-units). The values of ϕ/ϕ_0 are given in the following table.

TABLE III

Cross-section for Compton scattering in units ϕ_0 for various primary energies

γ	0·05	0·1	0·2	0·33	0·5	1	2	3	
ϕ/ϕ_0	0·91	0·84	0·74	0·63	0·56	0·43	0·31	0·254	
γ	5	10	20	50	100	200	500	1,000	
ϕ/ϕ_0	19	12·3	7·54	3·76	2·15	1·22	0·56	0·30	$\times 10^{-2}$

An *experimental test* of the theory is provided by measurements of the total absorption coefficient of X-rays or γ-rays in various materials. This absorption coefficient τ per cm. is given by

$$\tau = NZ\phi, \tag{56}$$

where N represents the number of atoms per cm.^{3} and Z the nuclear charge, i.e. the number of electrons per atom. *The absorption coefficient is proportional to the total number of electrons NZ per cm.*^{3} In (56) we have of course assumed that the primary radiation is so hard that all electrons can be considered as free. To obtain τ one has to multiply the values given in Table III or in Fig. 13 by $NZ\phi_0$. This quantity is given in the appendix for various substances.

For a comparison of the theory with experiment one has, however, to remember two points:

(1) For X-rays the total absorption is not only due to scattering. We have seen in § 13 that the *photoelectric effect* gives also a strong absorption which, however, decreases rapidly with the primary energy. To compare the photoelectric absorption with the scattering we have also plotted in Fig. 13 (dotted curves) the photoelectric cross-section ϕ_K of the K-shell (multiplied by 5/4, see p. 127) as computed in § 13 in the same units. (Since ϕ_K refers to a whole atom we have plotted $\phi_K/Z\phi_0$.) We see, for instance, that for carbon the photoelectric absorption is much greater than the scattering for a wave-length $\lambda > 500$ X.U., whereas for $\lambda < 300$ X.U. the scattering only is appreciable.

(2) On the other hand, for γ-rays, the absorption, at least for heavy

elements, is largely due to *pair production*, as we shall see in § **20**, Chapter IV. The pair production gives rise to an 'excess scattering' which is the larger the higher are the primary energy and the atomic number Z. For carbon, however, the pair production is negligible for $k_0 < 10\mu$.

For a comparison with the theory, therefore, we have made use only of those measurements for which the photoelectric absorption and the pair production is small compared with the absorption due to scattering. In Fig. 13 the experimental results have been plotted for three regions of wave-length:

(1) for X-rays of the wave-length 100–300 X.U.† (scattered by carbon);

(2) for hard X-rays $\lambda \simeq$ 20–50 X.U.‡ (carbon and aluminium);

(3) for the ThC″ γ-radiation of wave-length $\lambda = 4{\cdot}7$ X.U.‖ (carbon).

For the first region the (theoretical) photoelectric absorption, which is about 10 per cent. of the total absorption, has been subtracted. For the second and third regions no corrections were necessary.

The experiments fit the theoretical curve excellently. We may therefore consider the *Klein-Nishina formula as proved*, at least for energies up to $10mc^2$ Since other relativistic wave equations deviate appreciably in the region in question from the Klein-Nishina formula, we can consider the experimental results also as decisive evidence for Dirac's wave equation and against, for instance, the Klein-Gordon equation.

We shall return to the question of the absorption of γ-rays in § **22**.

For all these three regions the electrons are practically free and the coherent scattering is entirely negligible. This would, of course, no longer be true for softer radiation, for which the binding of the electrons has to be taken into account.†† As we have seen in § **14**

† C. W. Hewlett, *Phys. Rev.* **17** (1921), 284; S. J. M. Allen, ibid. **27** (1926), 266; **28** (1926), 907.

‡ J. Read and C. C. Lauritsen, *Phys. Rev.* **45** (1934), 433.

‖ L. Meitner and H. Hupfeld, *Zs. f. Phys.* **67** (1930), 147; C. Y. Chao, *Phys. Rev.* **36** (1930), 1519; *Proc. Nat. Ac.* **16** (1930), 431; G. T. B. Tarrant, *Proc. Roy. Soc.* **128** (1930), 345.

†† The effect of binding on the Compton scattering has been computed by H. Casimir, *Helv. Phys. Acta*, **6** (1933), 287; W. Franz, *Zs. f. Phys.* **90** (1934), 623; G. Wentzel, ibid. **43** (1927), 1 and 779; F. Bloch, *Phys. Rev.* **46** (1934), 674; A. Sommerfeld, *Ann. Phys.* **29** (1937), 715; W. Franz, ibid. 721.

subsection 3, the binding changes the intensity of the scattered radiation and gives rise to, besides the Compton scattering, a coherent scattering (unshifted line). Furthermore, the shifted (Compton-) line becomes, for a given angle of scattering, broader. The maximum of the shifted line is displaced by a very small amount as compared with the value given by the Compton formula (4). (Bloch, loc. cit.)

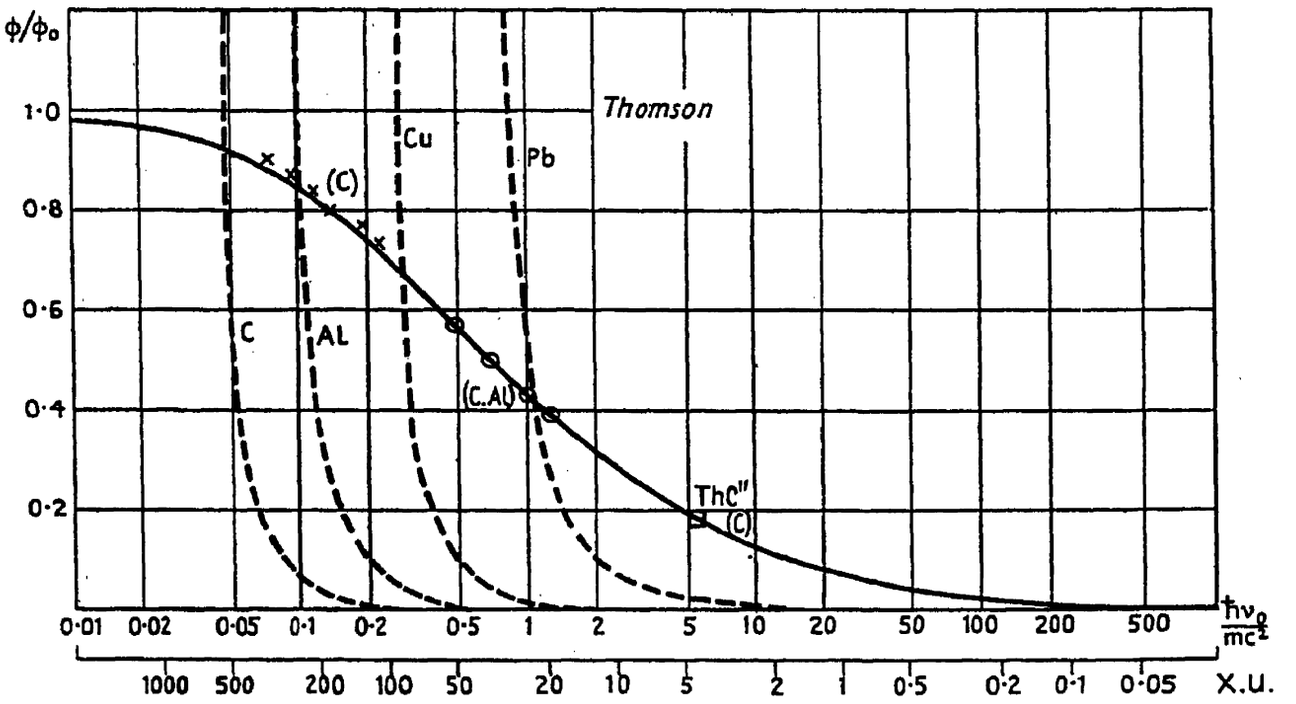

FIG. 13. Total cross-section for Compton scattering (Klein-Nishina formula) as a function of the primary frequency ν_0 (lower scale = primary wave-length). For comparison the cross-section for the photoelectric absorption per electron (dotted curves $\phi_K/Z\phi_0$). Measurements: × Hewlett, Allen (corrected for photo-absorption) in carbon; o Read and Lauritsen (carbon and aluminium); □ Meitner and Hupfeld, Chao (carbon).

C. PROCESSES OF THIRD AND HIGHER ORDERS IN e^2

Third-order processes are those in which three light quanta are simultaneously emitted or absorbed. However, they do not seem to be of great practical interest. We shall give a rough estimate of the order of magnitude of these and of higher processes in § **18**.

Processes of the same order in e are, however, those in which the emission (or absorption) of two quanta is replaced by the deflexion of an electron due to the Coulomb interaction with another particle (compare the classification given in § **10** subsection 4). This is for instance the case for the *Bremsstrahlung* (continuous X-ray emission).

The Bremsstrahlung is one of the most important processes affect-

ing the penetrating power of high-energy electrons by passing through matter. The phenomenon of cosmic ray *showers* rests on the Bremsstrahlung emitted by electrons with very high energies ($> 200mc^2$). An important step in the critical understanding of the radiation theory was reached when it was found that the theory gave correct results even for such high energies (§§ 23, 24). In the following section we shall, therefore, pay special attention to the high-energy region.

17. Bremsstrahlung

If an electron with primary energy E_0 (momentum $\mathbf{p}_0$) passes through the field of a nucleus (or atom) it is in general deflected. Since this deflexion always produces a certain acceleration, the electron, according to the classical theory, must emit radiation. In the quantum theory there will be a certain probability that a light quantum $\mathbf{k}$ is emitted, the electron making a transition to another state with energy E (momentum $\mathbf{p}$), where

$$E+k = E_0. \tag{1}$$

Since the nucleus is heavy compared with the electron, the momentum of electron plus light quantum is not in general conserved; the nucleus can take any amount of momentum. We therefore obtain a finite transition probability to any final state E, $\mathbf{p}$ which satisfies (1).

1. *Differential cross-section.*† The interaction causing the transition from the initial state A $(\mathbf{p}_0)$ to the final state F $(\mathbf{p},\mathbf{k})$ consists of two parts: (i) the interaction H of the electron with the radiation field giving rise to the emission of $\mathbf{k}$, and (ii) the interaction V of the electron with the atomic field. Thus the total interaction is given by

$$H' = H+V. \tag{2}$$

We shall treat V in this section as a perturbation in the same approximation as the interaction H with the field (compare § 10). This means an expansion of the transition probability in powers of e^2 (or Ze^2). The first approximation (Born's approximation) of this

† Rough estimate of the order of magnitude at high energies: W. Heitler, *Zs. f. Phys.* **84** (1933), 145. Exact calculations for all energies using Born's approximation: W. Heitler and F. Sauter, *Nature*, **132** (1933), 892; F. Sauter, *Ann. d. Phys.* **20** (1934), 404; H. Bethe and W. Heitler, *Proc. Roy. Soc.* **146** (1934), 83. Compare also for small energies: F. Sauter, *Ann. d. Phys.* **18** (1933), 486; N. F. Mott, *Proc. Camb. Phil. Soc.* **27** (1931), 255. In the latter paper it is the influence of the radiation forces on the scattering of electrons that has been studied.

expansion gives, however, correct results only if

$$2\pi\xi_0 \equiv 2\pi \frac{Ze^2}{\hbar v_0} \ll 1 \quad \text{and} \quad 2\pi\xi \equiv 2\pi \frac{Ze^2}{\hbar v} \ll 1, \tag{3}$$

where v_0, v represent the velocities before and after the collision. For light elements (3) is always satisfied if the primary energy is of relativistic order of magnitude except in a small frequency range where the electron has given nearly all its kinetic energy $E_0 - \mu$ to the light quantum and, after the process, has therefore a small velocity v. For small primary velocities our theory breaks down. For this case Sommerfeld has given an exact theory,† the results of which are quoted in subsection 2. Also for high energies $v \sim c$ but *heavy elements* we do not expect our theory to give very accurate results although the order of magnitude will certainly be correct. (For lead $Ze^2/\hbar c = 0{\cdot}6$.)

We consider first in this section the case of a pure Coulomb field $V = Ze^2/r$. The interaction H of the electron with the radiation only has non-vanishing matrix elements for transitions in which the momentum is conserved. On the other hand, the Coulomb interaction V only has matrix elements for which the state of the radiation field remains unchanged while the momentum of the electron may change by any amount. The transition $A \to F$ occurs, therefore, only by passing through an intermediate state. There are obviously two possible intermediate states:

I. $\mathbf{k}$ is emitted. The electron has a momentum $\mathbf{p}'$:

$$\mathbf{p}' = \mathbf{p}_0 - \mathbf{k}. \tag{4}$$

The transition $A \to$ I is caused by the term H of the interaction (2). In the transition to the final state F (due to the term V) the momentum of the electron changes from $\mathbf{p}'$ to $\mathbf{p}$.

II. The electron has momentum

$$\mathbf{p}'' = \mathbf{p} + \mathbf{k}, \tag{5}$$

chosen so that in the transition II $\to F$ $\mathbf{k}$ is emitted with conservation of momentum. The transition $A \to$ II is caused by the Coulomb interaction V

For all these transitions, either V or H makes a contribution to the

† A. Sommerfeld, *Ann. d. Phys.* **11** (1931), 257; O. Scherzer, ibid. **13** (1932), 137. Compare also: Y. Sugiura, *Phys. Rev.* **34** (1929), 858; *Scientif. Papers Inst. Phys. Chem. Research, Tokyo*, **11** (1929), 251; **13** (1930), 23; J. R. Oppenheimer, *Zs. f. Phys.* **55** (1929), 725; J. A. Gaunt, *Zs. f. Phys.* **59** (1930), 508.

matrix element. The matrix elements of H and V for the transitions in question can be taken immediately from § **10** eqq. (16) and (22):

$$H_{AI} = -e\sqrt{\left(\frac{2\pi\hbar^2c^2}{k}\right)}(u_0^* \alpha u'); \quad V_{IF} = \frac{Ze^2 4\pi\hbar^2c^2}{|\mathbf{p}'-\mathbf{p}|^2}(u'^* u)$$
$$V_{AII} = \frac{Ze^2 4\pi\hbar^2c^2}{|\mathbf{p}_0-\mathbf{p}''|^2}(u_0^* u''); \quad H_{IIF} = -e\sqrt{\left(\frac{2\pi\hbar^2c^2}{k}\right)}(u''^* \alpha u). \tag{6}$$

α represents the component of $\boldsymbol{\alpha}$ in the direction of polarization of $\mathbf{k}$.

The denominators in the matrix elements of V are equal by (4) and (5); we denote them by q^2, so that

$$\mathbf{q} = \mathbf{p}_0-\mathbf{p}'' = \mathbf{p}'-\mathbf{p} = \mathbf{p}_0-\mathbf{p}-\mathbf{k}. \tag{7}$$

$\mathbf{q}$ represents the total momentum transferred to the nucleus.

The two intermediate states I and II have multiplicity four, because in the intermediate state the electron may have either of two spin directions and positive or negative energy (cf. the theory of the Compton effect). Denoting by $\sum$ the summation over these four states with the same momentum $\mathbf{p}'$, $\mathbf{p}''$ the matrix element responsible for the transition $A \to F$ is given, according to § **9** eq. (43 b) by

$$H'_{AF} = \sum\left(\frac{H_{AI}V_{IF}}{E_A-E_I}+\frac{V_{AII}H_{IIF}}{E_A-E_{II}}\right), \tag{8}$$

where the energy differences E_A-E_I and E_A-E_{II} are, by (4) and (5):

$$E_A-E_I = E_0-k-E', \quad E'^2 = p'^2+\mu^2$$
$$E_A-E_{II} = E_0-E'', \quad E''^2 = p''^2+\mu^2. \tag{9}$$

The transition probability per unit time is given by § **9** eq. (42):

$$w = \frac{2\pi}{\hbar}|H'_{AF}|^2\rho_F, \tag{10}$$

where ρ_F represents the number of final states per energy interval dE_F. In the final state we have an electron with momentum $\mathbf{p}$ (energy E) and a light quantum $\mathbf{k}$.

Since $\mathbf{k}$ and $\mathbf{p}$ are independent of each other (in contrast to the Compton effect § **16** eq. (16)), the number of final states ρ_F is equal to the product of two density functions ρ_E and ρ_k for the electron and the light quantum. The energy interval dE_F, for a given k, can be equated to the energy interval dE of the electron. Thus we have

$$\rho_F = \rho_E\rho_k dk = \frac{pE\,d\Omega\,k^2\,d\Omega_k}{(2\pi\hbar c)^6}dk. \tag{11}$$

Inserting (11) in (10), we obtain the probability w that a light quantum with an energy between k and $k+dk$ is emitted in the element of solid angle $d\Omega_k$ and that the electron is deflected into the element $d\Omega$.

Dividing w by the velocity of the incident electron cp_0/E_0, we obtain from (6), (8), (9), (10), (11) the differential cross-section for the process in question:

$$d\phi = \frac{Z^2e^4}{137\pi^2}\frac{pEE_0}{p_0q^4}d\Omega d\Omega_k\left|\sum\left(\frac{(u_0^*\alpha u')(u'^*u)}{E-E'}+\frac{(u_0^*u'')(u''^*\alpha u)}{E_0-E''}\right)\right|^2. \quad (12)$$

(12) refers to given spin directions before and after the process. Since we are not interested in the direction of the spin, we must carry out a summation S over the spin directions of the final state and must average ($\frac{1}{2}S_0$) over the spin directions in the initial state.

We do not wish to discuss the polarization† of the radiation emitted; we shall therefore sum (12) also over both directions of polarization of **k**. All these summations, including that occurring in (12) ($\sum$), can be carried out by exactly the same method as that used in the theory of the Compton effect, § **16** subsection 3. Since the methods used for the computation of the matrix elements (12) are described there we shall give here the result only. The differential cross-section becomes

$$d\phi = \frac{Z^2e^4}{2\pi 137}\frac{dk}{k}\frac{p}{p_0}\frac{\sin\theta\,d\theta\sin\theta_0\,d\theta_0\,d\phi}{q^4}\times$$

$$\times\left\{\frac{p^2\sin^2\theta}{(E-p\cos\theta)^2}(4E_0^2-q^2)+\frac{p_0^2\sin^2\theta_0}{(E_0-p_0\cos\theta_0)^2}(4E^2-q^2)-\right.$$

$$-2\frac{pp_0\sin\theta\sin\theta_0\cos\phi}{(E-p\cos\theta)(E_0-p_0\cos\theta_0)}(4E_0E-q^2+2k^2)+$$

$$\left.+2k^2\frac{p^2\sin^2\theta+p_0^2\sin^2\theta_0}{(E-p\cos\theta)(E_0-p_0\cos\theta_0)}\right\}, \quad (13)$$

where θ, θ_0 are the angles between k and $\mathbf{p}$, $\mathbf{p}_0$ respectively and ϕ is the angle between the planes (**pk**) and ($\mathbf{p}_0$**k**). The denominators $(E-p\cos\theta)$, $(E_0-p_0\cos\theta_0)$ have their origin in the resonance denominators of (12):

$$E_0^2-E''^2 = 2k(E-p\cos\theta),$$
$$E^2-E'^2 = -2k(E_0-p_0\cos\theta_0).$$

† In the non-relativistic case the polarization of the radiation is discussed by Sommerfeld (loc. cit.).

q is a function of the angles, and according to (7) it is given by

$$q^2 = p_0^2+p^2+k^2-2p_0 k\cos\theta_0+2pk\cos\theta- \\ -2p_0 p(\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos\phi). \quad (14)$$

(13) gives the probability that a quantum **k** is emitted in a direction forming an angle θ_0 with the direction of the primary electron and that the electron is scattered in a direction with polar angles θ, ϕ referred to **k**. Before discussing the angular distribution given by (13), we integrate the differential cross-section over all angles and obtain the total cross-section for the emission of a quantum k with an energy between k and $k+dk$. It will be convenient to represent the cross-section as a function of the ratio of k to the initial *kinetic* energy $E_0-\mu$ (i.e. of $k/(E_0-\mu)$, which then ranges from 0 to 1). We therefore define the cross-section ϕ_k for the emission of a quantum k in the range $dk/(E_0-\mu)$ by

$$\phi_k d\frac{k}{E_0-\mu} = \int d\phi d\Omega d\Omega_k. \quad (15)$$

The integration of (15) over the angles is elementary but rather tedious. It yields

$$\phi_k d\frac{k}{E_0-\mu} = \bar{\phi}\frac{dk}{k}\frac{p}{p_0}\left\{\frac{4}{3}-2E_0E\frac{p^2+p_0^2}{p^2p_0^2}+\mu^2\left(\frac{\epsilon_0 E}{p_0^3}+\frac{\epsilon E_0}{p^3}-\frac{\epsilon\epsilon_0}{p_0 p}\right)+\right. \\ +L\left[\frac{8}{3}\frac{E_0E}{p_0p}+\frac{k^2}{p_0^3p^3}(E_0^2E^2+p_0^2p^2)+\right. \\ \left.\left.+\frac{\mu^2k}{2p_0p}\left(\frac{E_0E+p_0^2}{p_0^3}\epsilon_0-\frac{E_0E+p^2}{p^3}\epsilon+\frac{2kE_0E}{p^2p_0^2}\right)\right]\right\}, \quad (16)$$

where the following abbreviations have been used

$$L = \log\frac{p_0^2+p_0p-E_0k}{p_0^2-p_0p-E_0k} = 2\log\frac{E_0E+p_0p-\mu^2}{\mu k} \\ \epsilon_0 = \log\frac{E_0+p_0}{E_0-p_0} = 2\log\frac{E_0+p_0}{\mu}, \quad \epsilon = 2\log\frac{E+p}{\mu}, \quad \bar{\phi} = \frac{Z^2r_0^2}{137}. \quad (16')$$

$\bar{\phi}$ is a suitable unit in which to express the cross-section for Bremsstrahlung and similar processes. It is proportional to the *square of the nuclear charge.*

(16) is valid for all energies except very small ones for which Born's approximation (3) cannot be applied.

2. *Continuous X-ray spectrum.* The first application that we shall make of our theory is to the *continuous X-ray spectrum.* Here in

practice the primary energies are small compared with the rest energy of the electron μ, i.e. we have a *non-relativistic problem.*

Equating E_0 and E to μ and neglecting all the p's and k compared with μ, we obtain for the differential cross-section (13) simply

$$d\phi = \frac{2Z^2e^4}{\pi 137}\frac{dk}{k}\frac{p}{p_0}\frac{\sin\theta\, d\theta \sin\theta_0\, d\theta_0\, d\phi}{q^4}\times$$

$$\times\{p^2\sin^2\theta + p_0^2\sin^2\theta_0 - 2pp_0\sin\theta\sin\theta_0\cos\phi\}. \quad \text{N.R.} \quad (17)$$

Since $k = (p_0^2 - p^2)/2\mu$ is small compared with p_0, we may write for q^2, according to (14),

$$q^2 = p^2 + p_0^2 - 2pp_0(\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos\phi)$$
$$= (\mathbf{p}_0 - \mathbf{p})^2. \quad \text{N.R.} \quad (17')$$

$\mathbf{q}$ does not then depend upon the direction of $\mathbf{k}$.

For a given direction of deflexion (angle between $\mathbf{p}_0$ and $\mathbf{p}$) the maximum intensity is emitted perpendicular to the plane of motion of the electron ($(\mathbf{p}_0\mathbf{p})$-plane). This corresponds to the classical theory, where the maximum intensity is emitted perpendicular to the acceleration. The intensity emitted in a certain direction θ_0 is given by integrating (17) over all directions of $\mathbf{p}$.

The total emitted intensity of frequency k is given by (15). In the non-relativistic case we obtain simply

$$\phi_k\, d\left(\frac{k}{T_0}\right) = \bar{\phi}\frac{16}{3}\frac{dk}{k}\frac{\mu^2}{p_0^2}\log\frac{p_0+p}{p_0-p} = \bar{\phi}\frac{8}{3}d\left(\frac{k}{T_0}\right)\frac{\mu}{k}\log\frac{\{\sqrt{T_0}+\sqrt{(T_0-k)}\}^2}{k},$$

N.R. (18)

where $T_0 = E_0 - \mu = p_0^2/2\mu$ represents the kinetic energy of the primary electron. (18) shows that the probability for the emission of a quantum k decreases roughly as $1/k$. At the short wave-length limit $k = p_0^2/2\mu$, ϕ_k vanishes (see below), whereas for very long waves the intensity $k\phi_k$ diverges logarithmically. This, however, is only true in a pure Coulomb field and in Born's approximation. As we shall see in subsection 3, for a screened field $k\phi_k$ tends to a finite value for $k \to 0$. On the other hand, we shall see (eq. (19)) that ϕ_k remains finite for $k = p_0^2/2\mu$ in an exact theory.

The formulae (17), (18) are only valid as long as the condition (3) for the applicability of Born's approximation is satisfied. For very small energies it is no longer legitimate to compute the matrix elements of V by inserting plane waves for the wave functions of the electron. One has then to use the exact wave functions of the con-

tinuous spectrum (§ 9 (10)). This has been done by Sommerfeld (loc. cit.). We confine ourselves to quoting his results:

The angular distribution remains the same; the total intensity given by (17) and (18) has to be multiplied by the factor

$$f(\xi,\xi_0)=\frac{2\pi\xi 2\pi\xi_0}{(1-e^{-2\pi\xi})(e^{2\pi\xi_0}-1)},\qquad \xi=\frac{Ze^2}{\hbar v},\qquad \xi_0=\frac{Ze^2}{\hbar v_0}. \tag{19}$$

The factor (19) can be understood from formula (11) § 9 giving the wave function of the continuous spectrum† in the neighbourhood of the nucleus.

For high v_0 (i.e. for small ξ_0) the factor (19) deviates from the value 1 chiefly at the short wave-length limit. For $p\to 0$ (19) becomes infinite and, as (18) gave zero, the cross-section ϕ_k tends to a finite value. For small v_0 (large ξ_0) the factor (19) becomes smaller than 1 except in the immediate neighbourhood of the short wave-length limit.

In Fig. 14, p. 170, we have plotted the *intensity* $k\phi_k$ of the emitted radiation in units $E_0\bar{\phi}$ (in the non-relativistic case $E_0=\mu$) as a function of the ratio $k/(E_0-\mu)$. In the non-relativistic case (lowest curve) formula (18) (dotted part of the curve) gives an intensity distribution which is independent of the primary energy T_0. Approaching the region of hard quanta, the Sommerfeld factor (19) leads to a deviation from the curve calculated with Born's approximation, depending upon the primary energy and upon the nuclear charge Z (we disregard, of course, the factor Z^2 contained in the unit $\bar{\phi}$). In the graph the full curve refers to aluminium ($Z=13$) and $p_0/\mu=\frac{1}{2}$, i.e.

$$T_0/\mu=0{\cdot}125\quad(2\pi\xi_0=1{\cdot}2).$$

For heavy elements the deviation due to (19) is considerably larger. The intensity is then very much smaller than that given by the dotted curve (except for $k/T_0\sim 1$).

In the region of low-energy quanta also the 'exact' curve (19) is dotted, because in this region the intensity distribution will be altered by the screening.

For further discussion of the continuous X-ray spectrum, especially of the polarization and angular distribution and of the comparison with the experiment, the reader is referred to Sommerfeld's paper (loc. cit.).

3. *High energies, effect of screening.* At high energies the maximum in the angular distribution in the forward direction becomes more

† The first factor of (19) represents simply $|\psi|^2$ for the electron in the final state at the position of the nucleus. For the electron in the initial state the sign of ξ has to be reversed. In Born's approximation $|\psi|^2=1$ at the position of the nucleus.

and more pronounced. This can clearly be seen if we consider the extreme relativistic case E, $E_0 \gg \mu$. Then p_0 is very nearly equal to E_0 and the denominators $E_0 - p_0 \cos\theta_0$, etc., become very small for small angles θ_0. Also q has its minimum value for small angles θ_0 and θ. The electron and the light quantum are then both deflected (emitted) in the forward direction, the solid angle into which they are both projected being of the order $\Omega \sim mc^2/E_0$. This is a general feature of all quantum processes at very high energies. (Compare the photoelectric effect and the Compton effect. In the latter process the scattered quantum is emitted in the forward direction, if its energy k is large compared with μ.)

The frequency distribution is given by equation (16). In the extreme relativistic case E_0, $E \gg \mu$ this formula becomes

$$\phi_k d\left(\frac{k}{E_0}\right) = 2\phi d\left(\frac{k}{E_0}\right)\frac{E}{k}\left[-\frac{2}{3}+\frac{E_0^2+E^2}{E_0 E}\right]\left[2\log\frac{2E_0 E}{\mu k}-1\right]. \quad \text{E.R.} \quad (20)$$

For a given ratio k/E_0 the probability of emission increases roughly with the logarithm of E_0/μ. For small quanta $k \sim 0$ the intensity $k\phi_k$ diverges logarithmically.

The formula for the differential cross-section (13) and therefore all other formulae (16)–(20) have been derived under the assumption that the field of the nucleus is a *pure* Coulomb field. The question arises as to whether the *screening* of the Coulomb field due to the charge distribution of the outer electrons necessitates any important alterations. To decide this question, one would ask, in a classical treatment of our problem, whether the field is screened appreciably for those *impact parameters* r which give the main contribution to the effect. In the quantum theory, however, the idea of impact parameter has no exact meaning because the electron here is not represented by a point charge, but by a plane wave. Actually the averaging over all impact parameters is already contained in the integral representing the matrix element of V:

$$V_{IF} = \int \frac{e^{i(\mathbf{p}'-\mathbf{p},\mathbf{r})/\hbar c}}{r}\, d\tau = \int \frac{e^{i(\mathbf{q}\mathbf{r})/\hbar c}}{r}\, d\tau. \quad (21)$$

We can give a rough meaning to the idea of impact parameter by asking from which distances r the main contributions to the integral (21) arise. They obviously come from a distance

$$r \sim \hbar c/q. \quad (22)$$

For distances larger than (22) the contribution to (21) is small because the exponential function oscillates rapidly in a region where r is practically constant. For smaller distances $d\tau \sim r^2\,dr$ is small. We may therefore consider the quantity (22) as the most important *impact parameter*.

Now we have just seen that the differential cross-section (13) becomes very large if q is very small. For high primary energies, $E_0 \gg \mu$, the minimum value of q is given by

$$q_{\min} = p_0 - p - k \sim \frac{\mu^2 k}{2E_0 E}. \tag{23}$$

According to (22) therefore, we still obtain a large contribution to the cross-section from distances of the order

$$r_{\max} = \hbar c/q_{\min} = \frac{\hbar}{mc}\frac{2E_0 E}{\mu k} \sim \lambda_0 \frac{E_0 E}{\mu k}. \tag{24}$$

For a given ratio k/E_0, $r_{\max}$ is the larger the *higher* the primary energy. If k is of the order of magnitude of E_0, $r_{\max}$ becomes of the order of magnitude of the radius of the K-shell for $E_0 \sim 137\mu/Z$. We should expect, therefore, that just for *high energies* the *screening* of the Coulomb field by the outer electrons will lead to a *decrease* in the cross-section (a decrease, because the effective field is then less than the Coulomb field would be). For soft quanta k this will be the case, according to (24), for somewhat smaller energies. However, if k is not too small, the screening will only be effective in the extreme relativistic case $E_0 \gg \mu$.

We can get a rough idea of the effect of screening by considering the case where $r_{\max}$ is large compared with the atomic radius. We shall then call the *screening 'complete'*. For the atomic radius we may assume the value given by the Thomas-Fermi model:

$$a \sim a_0 Z^{-\frac{1}{3}} \sim 137\lambda_0 Z^{-\frac{1}{3}}. \tag{25}$$

(a_0 = Bohr's radius of the hydrogen atom.) If now $r_{\max}$ is large compared with a, we shall certainly obtain the right order of magnitude for the cross-section if we *replace the maximum impact parameter* $r_{\max}$ by the *atomic radius* a (25). In the formula (20) for the frequency distribution $r_{\max}$ occurs under the logarithm, and this fact determines the chief features of the distribution curves for very high energies and for quanta of low energy. This logarithm has now to be replaced by $\log(137Z^{-\frac{1}{3}})$. For a given value of k/E_0, ϕ_k will then tend to a finite

value as $E_0/\mu \to \infty$. Also, for low-energy quanta ($k \sim 0$), the intensity $k\phi_k$ tends to a finite value instead of diverging logarithmically.

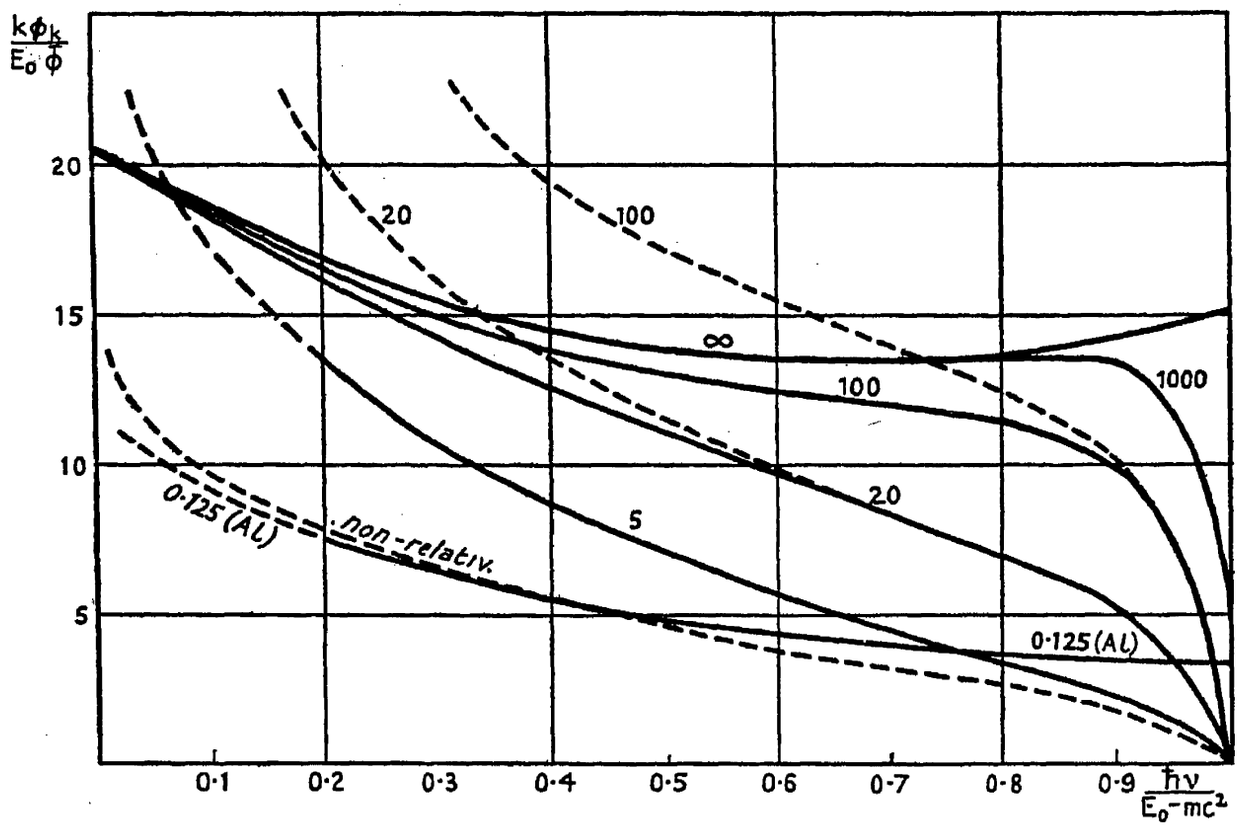

FIG. 14. Intensity distribution of the Bremsstrahlung as a function of $\hbar\nu/(E_0 - mc^2)$. The numbers affixed to the curves refer to the primary kinetic energy $E_0 - \mu$ in units of $\mu = mc^2$. The dotted portions are calculated using Born's approximation and neglecting the screening (formula (16)), and are valid for all elements. The full curves, where deviating from the dotted curves, are calculated for lead (except that for the non-relativistic case which is calculated for aluminium). The deviations represent (i) at high energies and for soft quanta the effect of screening, (ii) for small energies (non-relativistic curve) the effect of the Sommerfeld factor (19). Units: $\bar{\phi} = Z^2 r_0^2/137$.

Assuming a Thomas-Fermi model for the atom, the exact calculations† lead, for the case of complete screening, to the following formula for the frequency distribution:

$$\phi_k d\left(\frac{k}{E_0}\right) = 2\bar{\phi}\frac{dk}{E_0}\frac{E}{k}\left[\left(\frac{E_0^2+E^2}{E_0 E} - \frac{2}{3}\right)2\log(183Z^{-\frac{1}{3}}) + \frac{2}{9}\right] \quad \text{E.R.} \quad (26)$$

$$\left(\text{for } \frac{E_0 E}{\mu k} \gg a/\lambda_0 = 137Z^{-\frac{1}{3}}\right).$$

(26) agrees roughly with what we should expect.

If the screening is not complete, the theory gives a continuous transition from formula (20) to formula (26).

† H. Bethe, *Proc. Camb. Phil. Soc.* **30** (1934), 524; H. Bethe and W. Heitler, loc. cit.

Since ϕ_k is roughly proportional to E_0/k we have plotted in Fig. 14 the *intensity* $k\phi_k$ in units $E_0\bar{\phi}$ for various primary kinetic energies $E_0-\mu$. The dotted portions of the curves, where they deviate from the full curves, are calculated for the case where the screening is neglected (formulae (16), (20)). They are valid for all elements: Z is contained only in the quantity $\bar{\phi}$. The full curves are calculated for lead ($Z = 82$), except that for non-relativistic energies which is valid for aluminium; they approach the curve for complete screening $E_0 \sim \infty$ (formula (26)) in the region of low-energy quanta. Here, also, the non-relativistic curve would tend to a finite value if the screening were taken into account. In the region of high-energy quanta the screening can be neglected, so that here the full and dotted curves do not differ (curves valid for all Z). On the other hand, at the limit for high-energy quanta ($k = E_0-\mu$) the curves would probably tend to a finite value if the correct wave functions of the continuous spectrum were used, as is the case in Sommerfeld's exact theory for non-relativistic energies.

For light elements the screening becomes less appreciable. For heavy elements the use of Born's approximation leads to some error even for medium and soft quanta (compare subsection 2).

As we see from the graph the intensity distribution is roughly uniform over the whole frequency range.

4. *Energy loss.* An appreciable fraction of the energy lost by an electron in passing through matter is due to the fact that while passing through the field of an atom it may emit a light quantum with energy of the order of magnitude of its own kinetic energy. The average energy lost in one collision may be obtained by integrating the intensity $k\phi_k$ over all frequencies from 0 to $E_0-\mu$. The average energy lost per cm. path is given by

$$-\frac{dE_0}{dx} = N \int_0^1 k\phi_k \, d\left(\frac{k}{E_0-\mu}\right), \tag{27}$$

where N represents the number of atoms per cm.^{3} Fig. 14 shows that the area under the curves for all energies is of the same order of magnitude. Since in Fig. 14 we have plotted the cross-section divided by E_0, the energy lost per cm. will be roughly proportional to the primary energy E_0 for high energies $E_0 \gg \mu$, and constant ($E_0 = \mu$) for small kinetic energies $E_0-\mu \ll \mu$. It is therefore convenient to define a

cross-section, ϕ_{rad}, for the energy lost by radiation:

$$-\frac{dE_0}{dx} = NE_0\,\phi_{\text{rad}}$$

$$\phi_{\text{rad}} = \frac{1}{E_0}\int_0^1 k\phi_k\, d\left(\frac{k}{E_0-\mu}\right). \tag{28}$$

The integration (28) can be carried out if for ϕ_k the formulae obtained by using Born's approximation are inserted. From (16) we obtain,† after an elementary but rather tedious calculation,

$$\phi_{\text{rad}} = \bar{\phi}\left\{\frac{12E_0^2+4\mu^2}{3E_0p_0}\log\frac{E_0+p_0}{\mu} - \frac{(8E_0+6p_0)\mu^2}{3E_0p_0^2}\left(\log\frac{E_0+p_0}{\mu}\right)^2 - \right.$$
$$\left. -\frac{4}{3}+\frac{2\mu^2}{E_0p_0}F\left(\frac{2p_0(E_0+p_0)}{\mu^2}\right)\right\}, \tag{29}$$

where the function F is defined by the integral

$$F(x) = \int_0^x \frac{\log(1+y)}{y}\,dy. \tag{29'}$$

For small x the function F can be expanded in a power series:

$$F(x) = x-\frac{x^2}{4}+\frac{x^3}{9}-\frac{x^4}{16}+\ldots. \tag{30}$$

For large x one can use the (exact) formula:

$$F(x) = \tfrac{1}{6}\pi^2+\tfrac{1}{2}(\log x)^2-F(1/x). \tag{31}$$

From (29), (30), and (31) we obtain for the two limiting cases

$$\phi_{\text{rad}} = \tfrac{16}{3}\bar{\phi}, \qquad \text{N.R.} \tag{32}$$

$$\phi_{\text{rad}} = 4\left(\log\frac{2E_0}{\mu}-\frac{1}{3}\right)\bar{\phi}. \qquad \text{E.R.} \tag{33}$$

For small energies the average energy radiated is independent of the primary energy.‡ For high energies the ratio of the radiated energy to the initial energy increases logarithmically with E_0. This, however, is only true if the screening is neglected. In the case of complete screening we obtain from equation (26) by integrating

$$\phi_{\text{rad}} = \bar{\phi}\{4\log(183Z^{-\frac{1}{3}})+\tfrac{2}{9}\} \qquad \text{E.R.} \tag{34}$$
$$(\text{for } E_0 \gg 137\mu Z^{-\frac{1}{3}}).$$

ϕ_{rad} is then constant.

† G. Racah, *Nuovo cimento*, XI (1934), N. 7.

‡ Thus the ratio of the energy radiated per cm. path to the initial kinetic energy $E_0-\mu$ becomes very large for $E_0-\mu\to 0$, at least as long as Born's approximation holds.

The cross-section ϕ_{rad} for energy loss by emission of radiation is plotted in Fig. 15 in a logarithmic scale. Formula (33) for high energies with neglect of screening gives a straight line. The curves with screening are obtained from Fig. 14 by numerical integration. They tend to the asymptotic value given by equation (34).

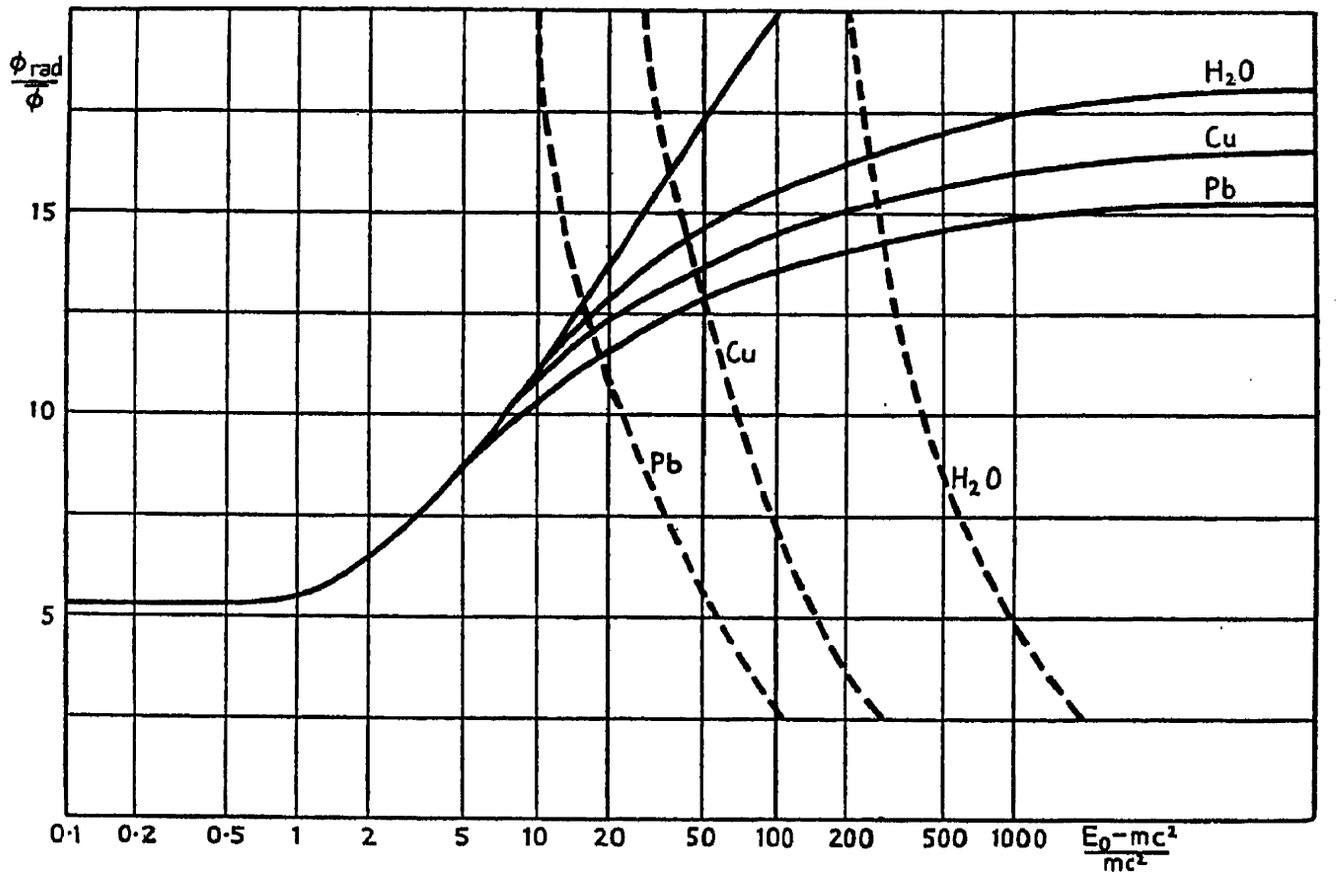

FIG. 15. Cross-section ϕ_{rad} for the energy loss of an electron per cm. path by radiation (ϕ_{rad} is defined in equation (28)). Units $\bar{\phi} = r_0^2 Z^2/137$. The straight line at high energies is calculated neglecting the effect of screening and is valid for all elements. The dotted curves show in the same units the energy loss by inelastic collisions for various materials.

The curves are quantitatively correct for light elements. For heavy elements we must bear in mind, however, that the use of Born's approximation leads to some error in the numerical values, but the order of magnitude is certainly correct even for lead.

The values of $\phi_{rad}/\bar{\phi}$ are given in the following table:

TABLE IV

Cross-section for the energy loss by radiation

$(E_0-mc^2)/mc^4$	0	1	2	5	10	50	100	∞
$\phi_{rad}/\bar{\phi}$	5·33	5·5	6·5	8·8	10·3 (Pb)	12·8 (Pb)	13·5 (Pb)	15·3 (Pb)

To give a rough idea of the practical significance of our results, in Fig. 15 we have plotted for comparison in the same units the

cross-section for the ordinary energy loss of an electron by inelastic collisions (ionization of atoms). The general question of the energy loss of particles in passing through matter will be considered in § **23** in context. Here we only mention that the energy loss per cm. by collisions is proportional to Z (whereas our effect is proportional to Z^2) and roughly independent of the energy. The ratio of the energy loss to the primary energy (which is plotted in Fig. 15) *decreases* therefore as $1/E_0$ whereas the same quantity for the energy loss by radiation *increases* logarithmically. We obtain, therefore, the striking result, that for *energies higher than a certain limit, the energy loss is almost entirely due to emission of radiation* and reaches a *value which is much higher than that due to ionization*. For lead this limit lies at about $20mc^2$, for water at $250mc^2$.

We shall return to this topic in § **23**, where we shall also compare our result with the experimental facts about cosmic radiation.

5. *Semi-classical deduction of the cross-section for the energy loss.* We have shown in the preceding section that the theory leads to a very high value for the energy loss due to emission of radiation. It therefore seems desirable to see, at least qualitatively, the origin of this result more clearly than these rather complicated calculations allow. This is all the more important since the large energy loss for energies $> 200\mu$ is fundamental for the interpretation of cosmic radiation (§§ **23**, **24**).

We shall see that the order of magnitude of ϕ_{rad} as given by (33) and (34) is actually a *classical* effect and can easily be deduced by a very rough classical treatment of our problem. Like all results obtained by the present quantum theory of radiation, the theory of Bremsstrahlung developed in this section represents, in the sense of the correspondence principle, nothing more than the quantum analogy of the classical theory.

We consider an electron passing through the Coulomb field of a nucleus at a distance r† from it. The energy E_0 of the electron is to be large compared with mc^2 and its velocity therefore nearly equal to c:

$$\gamma = \frac{E_0}{\mu} = \frac{1}{\sqrt{(1-\beta^2)}} \gg 1, \qquad \beta = v_0/c. \tag{35}$$

Let the direction of v_0 be along the z-axis.

† Cf. for this method C. F. v. Weizsäcker, *Zs. f. Phys.* **88** (1934), 612; E. J. Williams, *Phys. Rev.* **45** (1934), 729.

The Coulomb field causes an acceleration (deflexion) of the electron which itself gives rise to an emission of radiation. Since the formulae (§ **3** (10)) for the radiation by a fast electron are very complicated, it is convenient to consider the same process in a coordinate system in which the *electron* is at *rest*. In this Lorentz system the *nucleus* passes with the velocity $-v_0$ at the same distance r (r is not transformed because it is perpendicular to z). We denote by dashed symbols all quantities referred to this system of coordinates.

The field accompanying the nucleus is the Coulomb field transformed by a Lorentz transformation. The component of the electric field strength in the direction of r is therefore, according to § **2** (19), given by

$$E'_r = \frac{Ze}{r^2}\gamma. \tag{36}$$

We can assume that, at a distance r, the field of the nucleus at rest is roughly constant along the z-direction within a region r. In the Lorentz system, in which the electron is at rest, the extension ρ of the field is compressed in the z-direction. Therefore we have, by § **2** eq. (3),

$$\rho' = r\sqrt{(1-\beta^2)} = r/\gamma. \tag{37}$$

The acceleration due to the nucleus is equal to

$$b' = \frac{eE'_r}{m} = \frac{Ze^2}{mr^2}\gamma. \tag{38}$$

Since the electron is now at rest, the *energy radiated* per unit time is given by § **3** eq. (24):

$$S' = \frac{e^2}{c^3}b'^2 = \frac{Z^2e^6}{m^2c^3r^4}\gamma^2. \tag{39}$$

We obtain the total energy radiated by multiplying (39) by the time τ during which the collision takes place, τ, by (37), being roughly given by

$$\tau' = \rho'/c = r/\gamma c, \tag{40}$$

and hence

$$S'\tau' = \frac{Z^2e^6}{m^2c^4r^3}\gamma = \frac{Z^2r_0^2e^2}{r^3}\gamma. \tag{41}$$

This energy radiated has a roughly uniform angular distribution. Transforming back to the original Lorentz system in which the nucleus is at rest, we obtain the total energy radiated by a fast-moving electron passing at a distance r by using the transformation formulae § **2** eq. (44)† for the momentum energy vector of the

† Owing to the uniform angular distribution the momentum of the radiation can be neglected if we wish to know only the order of magnitude of the effect.

radiation field:

$$S\tau = S'\tau'\gamma = \frac{Z^2e^2r_0^2}{r^3}\gamma^2. \tag{42}$$

Since the probability of the electron passing at a distance between r and $r+dr$ is $\sim r\,dr$, we obtain, dividing by the primary energy E_0, the cross-section for the radiative energy loss

$$\phi_{\text{rad}} = \frac{1}{E_0}\int S\tau r\,dr = Z^2r_0^2e^2\gamma\frac{1}{\mu}\int\frac{dr}{r^2}. \tag{43}$$

The integration in (43) should be from 0 to ∞. This, however, is not allowed for the following reason: Up to this point, we have described the motion of the electron entirely classically. We must take into account the fact that in the quantum theory the electron is represented by a *wave packet*. If we wish the energy of the electron to be determined with an accuracy $\Delta E_0 \sim mc^2$ say,† the dimensions of the wave packet may not be smaller than

$$\Delta x = \frac{\hbar c}{\Delta p_0} \sim \frac{\hbar c}{\Delta E_0} = \frac{\hbar}{mc} = \lambda_0. \tag{44}$$

The classical formula (43) can only be applied if the field does not vary appreciably within the wave packet. Since the extension of the field was r/γ, we have

$$r/\gamma \gg \Delta x = \lambda_0. \tag{45}$$

(43) is therefore only correct for $r > \lambda_0\gamma$. Inserting $\lambda_0\gamma$ as the lower limit in the integral (43), the cross-section amounts to

$$\phi_{\text{rad}} = Z^2r_0^2e^2\gamma\frac{1}{\mu}\int_{\lambda_0\gamma}^{\infty}\frac{dr}{r^2} = Z^2r_0^2\frac{e^2}{\hbar c} = \bar{\phi}. \tag{46}$$

This gives actually the same order of magnitude for ϕ_{rad} as was obtained from the exact theory (equations (33), (34)).

Thus even in the region where the classical theory can be applied, we obtain the extraordinarily high value for the radiative energy loss. The results obtained in the preceding section are, therefore, in complete agreement with the correspondence principle.

We shall see that the application of the present theory to very high energies ($E_0 > 200mc^2$) leads to an understanding of the cosmic ray showers. We shall return to this question after having

† For smaller wave packets with $\Delta E_0 > \mu$, the number of incident particles is undetermined, because then an indefinite number of electron pairs can be present (see § 19).

discussed these results in the light of the experimental evidence (§§ **23**, **24**).

18. Multiple processes, higher approximations, self-energy of the electron.

In the preceding paragraphs we have studied the most important radiation processes (except those which are connected with positive electrons). We have found that the quantum theory of radiation applied to the interaction of radiation with an electron always leads in the *first approximation* to reasonable results, which can be understood from the point of view of the correspondence principle and which are in agreement with experiment. In §§ **23**, **24** we shall see that the agreement persists up to the highest energies known in cosmic radiation.

In this final section of the chapter we shall treat a few problems which go beyond the field in which the theory has hitherto been applied.

There is first of all the important question of the *higher approximations* in the interaction between light and matter (compare § **10** subsection 1). As we shall see in subsection 2 the theory leads here to entirely meaningless results.

Another question which we shall discuss here is that of the *multiple processes*, i.e., for instance, processes in which instead of a single light quantum k two quanta k_1 and k_2 with the same total energy $k_1+k_2 = k$ are emitted. These multiple processes are of interest for two reasons:

(1) The order of magnitude of their probability cannot be estimated unambiguously from the classical theory by an application of the correspondence principle. They are typical quantum effects. For a critical understanding of the present theory it is in any case important to have a clear idea of the results to which they lead.†

(2) At very high energies in cosmic radiation theoretical physics faces the striking phenomenon of the '*showers*'.‡ In these showers, together with photons, a large number of electron pairs are emitted

† In the optical region the simultaneous emission of two quanta has been studied by M. Mayer-Göppert, *Naturw.* **17** (1929), 932; *Ann. Phys.* **9** (1931), 273. The transition probability is very small. Here we are interested more in the region of high energies.

‡ P. M. S. Blackett and G. P. S. Occhialini, *Proc. Roy. Soc.* **139** (1933), 699, and (with J. Chadwick) **144** (1934), 235; C. D. Anderson, *Phys. Rev.* **44** (1933), 406; C. D. Anderson, R. A. Millikan, S. Neddermeyer, and W. Pickering, ibid. **45** (1934), 352.

by a single fast particle or light quantum in passing through matter. We know now that most of these showers are of a cascade nature and that the theory of Bremsstrahlung and pair production fully accounts for them (§ 24). From the experimental evidence it can, however, not wholly be excluded that true multiple processes might also occur. In any case, it is desirable to know whether the present quantum theory gives, for those multiple processes, transition probabilities of the order of magnitude observed in the showers. This seems at first sight to be possible for the following reason:

In the preceding section we have seen that at high energies of an incident particle the emission of Bremsstrahlung has a much larger probability than an inelastic collision. From the point of view of an expansion in powers of e the Bremsstrahlung represents a process of higher order (third) than the inelastic collisions (second order). One cannot therefore exclude the possibility that the multiple processes which are of still higher order in e might, at high energies, have a transition probability large enough to explain the shower phenomenon. This, however, we shall show not to be the case.

1. *Multiple processes.* As a simple but typical example we consider first a process which we may call 'double Compton scattering'.† A primary light quantum $\mathbf{k}_0$ is 'scattered' by a free electron initially at rest in such a way that instead of one scattered quantum $\mathbf{k}$ two quanta $\mathbf{k}_1$ and $\mathbf{k}_2$ are emitted. The transition probability for this process can be computed in exactly the same way as for the ordinary scattering, § 16.

The conservation laws in this case are expressed by the equations

$$\mathbf{k}_0 = \mathbf{k}_1+\mathbf{k}_2+\mathbf{p}, \tag{1a}$$

$$k_0+\mu = k_1+k_2+E. \tag{1b}$$

From (1) it follows that, if $k_0 < \mu$, both quanta k_1 and k_2 are of the order k_0:

$$k_1, k_2 \sim k_0+O\left(\frac{k_0^2}{\mu}\right), \tag{2}$$

where $O(x)$ denotes a term at most of the order x. If, on the other hand, $k_0 \gg \mu$ and if both quanta k_1, k_2 are scattered through large angles, then

$$k_1, k_2 \sim \mu; \qquad p, E \sim k_0. \tag{3}$$

† W. Heitler and L. Nordheim, *Physica*, 1 (1934), 1059.

The transition from the initial state A ($\mathbf{k}_0, \mathbf{p}_0 = 0$) to the final state F ($\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}$) can occur only by passing through *two subsequent intermediate states*, for instance:

I. $\mathbf{k}_0$ absorbed. The electron has a momentum

$$\mathbf{p}' = \mathbf{k}_0. \tag{4}$$

II. $\mathbf{k}_1$ emitted. The electron has a momentum

$$\mathbf{p}'' = \mathbf{k}_0 - \mathbf{k}_1. \tag{5}$$

Other pairs of intermediate states are obtained by permuting the order in which the three quanta are emitted or absorbed. There exist altogether six pairs of intermediate states.

The matrix element which determines the transition probability $A \to F$ is now, according to § 9 eq. (43 c), given by

$$H_{AF} = \sum \frac{H_{A\,\mathrm{I}}\, H_{\mathrm{I\,II}}\, H_{\mathrm{II}\,F}}{(E_A - E_{\mathrm{I}})(E_A - E_{\mathrm{II}})}, \tag{6}$$

where the summation $\sum$ is over all six intermediate states, the spin directions, and the sign of the energy of the electron in the intermediate states. The energy differences in the denominator of (6) are, according to (4) and (5),

$$\begin{aligned} E_A - E_{\mathrm{I}} &= k_0 + \mu - E'; & E'^2 &= p'^2 + \mu^2; \\ E_A - E_{\mathrm{II}} &= k_0 + \mu - k_1 - E''; & E''^2 &= p''^2 + \mu^2. \end{aligned} \tag{7}$$

The summation $\sum$ in (6) can be carried out in the same way as in § 16. The transition probability is proportional to $|H_{AF}|^2$. Taking the summation over the spin directions of the electron in the final state and the average value in the initial state, we obtain, in exact analogy with § 16 eq. (28) (dropping all numerical factors),

$$\begin{aligned} &\tfrac{1}{2}\mathrm{S}\mathrm{S}_0 |H_{AF}|^2 \\ &= \frac{(e\hbar c)^6}{k_0 k_1 k_2} \frac{1}{E\mu^2} \left[\mathrm{Sp}\, \frac{\alpha_0 K' \alpha_1 K'' \alpha_2 (H+E) \alpha_2 K'' \alpha_1 K' \alpha_0 (1+\beta)}{k_0^2 [\mu(k_0 - k_1) - k_0 k_1 (1 - \cos\theta_1)]^2} + \ldots \right], \end{aligned} \tag{8}$$

where θ_1 represents the angle between $\mathbf{k}_0$ and $\mathbf{k}_1$ and

$$\begin{aligned} K' &= \mu(1+\beta) + k_0 + (\boldsymbol{\alpha}\mathbf{k}_0), \\ K'' &= \mu(1+\beta) + k_0 - k_1 + (\boldsymbol{\alpha}, \mathbf{k}_0 - \mathbf{k}_1). \end{aligned} \tag{9}$$

For an estimate of the order of magnitude it is sufficient to consider the first term in (8) arising from the first pair of intermediate states. All other terms are of the same order of magnitude.†

† It can be shown that the different intermediate states do not 'interfere' so as to reduce the order of magnitude of the result. The order of magnitude of the whole matrix element is given by the first term of (8).

Thus the differential cross-section is given by (compare § **16** (29)

$$d\phi \sim \frac{1}{\hbar c}|H_{AF}|^2 \frac{k_1^2 k_2^2\, dk_2}{(\hbar c)^6} = \frac{r_0^2}{137}\frac{k_1 k_2\, dk_2}{k_0 E}[\mathrm{Sp}\,(8)]. \tag{10}$$

In (10) we have also dropped the factor $\partial E_F/\partial k$ (see § **16** eq. (16)) because it is of the order of magnitude 1 for all energies.

For the evaluation of the spur in (8) we assume that k_1 and k_2 are always of the same order, because this is the only case of interest. We consider the two cases $k_0 \ll \mu$ and $k_0 \gg \mu$:

(i) $k_0 \ll \mu$. For $H+E$ we can write, according to (1),

$$H+E = E+\beta\mu+(\boldsymbol{\alpha}\mathbf{p}) = \mu(1+\beta)+(\boldsymbol{\alpha}\mathbf{p})+\frac{p^2}{2\mu}$$
$$= \mu(1+\beta)+(\boldsymbol{\alpha}\mathbf{p})+O\left(\frac{k_0^2}{\mu}\right). \tag{11}$$

The largest contribution to the spur (8) will be from those terms which are proportional to μ. Since, however, according to § **16** eqq. (31) and (33),

$$\begin{gathered}(1+\beta)\alpha(1+\beta) = (1-\beta^2)\alpha = 0,\\ (1+\beta)\alpha(\boldsymbol{\alpha}\mathbf{p})\alpha(1+\beta) = 0,\end{gathered} \tag{12}$$

and since by (9) and (11) μ occurs only in the form $\mu(1+\beta)$, only one term $\mu(1+\beta)$ of $H+E$ or K' or K'' gives a contribution to the spur. The numerator of the spur is therefore of the order μk_0^4. The denominator is of the order $\mu^2 k_0^4$ and therefore

$$\mathrm{Sp}\,(8) \sim 1/\mu. \tag{13}$$

Inserting (13) in (10) we obtain for the cross-section

$$\phi \sim \frac{r_0^2}{137}\left(\frac{k_0}{\mu}\right)^2 \qquad (k_0 \ll \mu). \qquad \text{N.R.} \quad (14)$$

(ii) $k_0 \gg \mu$. In this case $H+E$, K', K'' according to (1) are all of the order k_0. The numerator of Sp (8) is therefore of the order k_0^5 whereas the denominator is $\sim \mu^2 k_0^4$ and therefore

$$\mathrm{Sp}\,(8) \sim k_0/\mu^2.$$

For large angles of scattering we have from (1) $E \sim k_0$; $k_1, k_2 \sim \mu$. Thus the cross-section amounts to

$$\phi \sim \frac{r_0^2}{137}\frac{\mu}{k_0} \qquad (k_0 \gg \mu). \qquad \text{E.R.} \quad (15)$$

Comparing our results (14), (15) with the corresponding formulae

(54) and (55) of § **16** for single scattering, we obtain the following scheme for the order of magnitude of the cross-section (units r_0^2)

	$k_0 \ll \mu$	$k_0 \gg \mu$
single scattering	1	μ/k_0
double scattering	$k_0^2/137\mu^2$	$\mu/137k_0$

For small energies, $k_0 \ll \mu$, the double process is thus extremely rare. Its probability differs from that for the single process by a factor $k_0^2/137\mu^2$. It tends to zero in the transition to the classical theory because $k_0^2/137 = e^2\nu_0^2\hbar/c$ is proportional to $\hbar$.

At high energies, $k_0 \gg \mu$, the double scattering is comparatively more likely. Its probability, however, is still *smaller by a factor* 1/137 than that of the single process.

The results obtained can easily be generalized. For instance, one can show that the probability of an electron passing through the Coulomb field of a nucleus emitting two quanta instead of one (double Bremsstrahlung) is, at high energies, 137 times smaller than that for the emission of one quantum. The cross-section for the simultaneous emission of a '*photon shower*' consisting of n-quanta by a multiple Bremsstrahlung process is at high energies (i.e. if the electron and *all* quanta have energies $E_0 \gg mc^2$) of the order

$$\phi_{\text{rad}}^{(n)} = \frac{Z^2 r_0^2}{137^n}. \qquad \text{E.R.} \quad (16)$$

It decreases with n as the nth power of 1/137. Multiple processes are therefore *rare* events and the cosmic ray showers are not due to them. The electron pairs in a shower are emitted step-wise one after the other as will be explained in § **24**.

We may note that the theory of the *meson*, a new kind of particle discovered in cosmic radiation (§ **23**, subsection 5), leads one to expect that multiple processes involving mesons are more frequent. In this book, we are not, however, concerned with mesons.

2. *Transverse self-energy of the electron.*† While for the multiple processes the theory leads at least to reasonable results, this is not the case for the higher approximations of any quantum process.

We shall study the higher approximations of the interaction between an electron and light through a simple example. According

† I. Waller, *Zs. f. Phys.* **62** (1930), 673; J. R. Oppenheimer, *Phys. Rev.* **35** (1930), 461; L. Rosenfeld, *Zs. f. Phys.* **70** (1931), 454.

to § 10 the interaction energy is given by

$$H = -e(\alpha\mathbf{A}) = -e\sum_{\lambda}[q_\lambda(\alpha\mathbf{A}_\lambda)+q_\lambda^*(\alpha\mathbf{A}_\lambda^*)]. \tag{17}$$

Hitherto we have considered the interaction H as causing *transitions* of the unperturbed system between two quantum states. But it is obvious that the interaction H will also make a certain *contribution* to the total *energy* of the system. It is this change of the total energy of the system which we shall now calculate.

In the classical theory the average value of H vanishes in the first approximation:

$$\overline{H} = -\overline{e(\alpha\mathbf{A})} = 0, \tag{18}$$

since $\mathbf{A}$ is composed of harmonically oscillating waves and the average value of the amplitude of each wave is equal to zero.

In the quantum theory the average value of the interaction energy H in a certain unperturbed state is given, in the first approximation, by the diagonal matrix elements of H referring to this state. If the number of light quanta n_λ of each radiation oscillator λ is given, the *diagonal* matrix elements of each q_λ vanish (according to § 10 eq. (4c)). Therefore, in the first approximation the average value of H vanishes also in the quantum theory:

$$\overline{H^{(1)}} = 0. \tag{19}$$

This result corresponds to the classical one (18). It is certainly correct as are all results which have been obtained from the quantum theory of radiation in the first approximation.

Let us now consider the *second approximation*. According to a well-known formula of quantum mechanics the second approximation of the interaction energy H is given by (§ 9 eq. (43b))

$$W = \overline{H^{(2)}} = \sum_{n'}\frac{H_{nn'}H_{n'n}}{E_n-E_n'}, \tag{20}$$

where the summation $\sum$ is over all intermediate states n'. We consider the case of a free electron at rest. We may also assume that in the state n for which we compute the average energy no light quanta are present. According to § 10, $H_{nn'}$ is then different from zero if in the state n' one light quantum $\mathbf{k}_\lambda$ belonging to a radiation oscillator λ is present, and if the electron has a momentum $\mathbf{p}' = -\mathbf{k}_\lambda$.

Inserting for $H_{nn'}$ the matrix elements § 10 eq. (16) we obtain

$$W = 2\pi e^2\hbar^2c^2\sum\frac{1}{k}\frac{(u^*\alpha u')(u'^*\alpha u)}{\mu-E'-k}, \tag{21}$$

from the following consideration: In the classical theory the higher approximations of the self-force, although they depend upon the structure of the electron, vanish if the radius of the electron is put zero ($\bar{r}_0 \to 0$). The higher approximations of the interaction H correspond to these higher approximations of the classical self-force. For a point electron they should therefore vanish. Since, on the contrary, they diverge, this divergence can only be an effect of the quantization.

The divergence of the second approximation is a general feature of the theory. Also the transition probability w of all radiation processes diverges in the second approximation, whereas, as we have seen in this chapter, the first approximation gives correct results in all cases examined.

As a formal criterion, we may say that the theory gives in general non-converging results in cases where *the number of intermediate states is infinite.* As we have seen, in our case the summation over this infinite number of states does not converge.

The present difficulty arises obviously from the *large quanta* k. One might think that it could be avoided if we assume that the theory breaks down at high energies and that for high energies the interaction of light with an electron is actually much smaller than given by the present theory. If we postulate that the self-energy of the electron cannot be larger than mc^2 we must, according to (23), cut off the short waves at a limit

$$k_m = mc^2\sqrt{(2\pi 137)} \quad \text{or} \quad \lambda_m = \lambda_0/\sqrt{(2\pi 137)} = r_0\sqrt{(137/2\pi)}, \quad (24)$$

i.e. at a wave-length which is $\sqrt{(137/2\pi)}$ times larger than the electronic radius.† It is, however, doubtful whether this limit has any physical meaning. Weisskopf‡ has shown that the transverse self-energy diverges only logarithmically if one takes into account the modifications of the theory arising from the existence of positive electrons (Chapter IV). It is indeed easy to see that, if instead of the transi-

† If the electron has a momentum p (energy E), the formula (23) for the transverse self-energy has to be replaced by

$$W = \frac{1}{137\pi E}\Big[\int k\,dk - \Big(E - \frac{\mu^2}{2p}\log\frac{E+p}{E-p}\Big)\int dk\Big].$$

Disregarding the second term which diverges only linearly in order to limit W to a value $\sim E$, we must cut off all waves with a quantum energy larger than

$$E\sqrt{(2\pi 137)} = k_m E/\mu.$$

This limit is obtained directly by a Lorentz transformation from the limit k_m quoted in the text (eq. (24)).

‡ V. Weisskopf, *Zs. f. Phys.* **90** (1934), 817.

IV

RADIATION PROCESSES CONNECTED WITH POSITIVE ELECTRONS

19. General theory of the positive electron

1. *Introduction.* Dirac's relativistic wave equation for a free electron has, according to § 9, solutions corresponding to states in which the *energy is negative.* These negative energy states arise from the fact that, in the theory of relativity, the energy of a free particle is given by a square root

$$E = \pm\sqrt{(p^2+\mu^2)}. \tag{1}$$

Thus, for any given value of p the sign of E can be either positive or negative.

The existence and properties of these negative energy states give rise to serious difficulties. The acceleration of such an electron due to an external force is in the opposite direction to the force. Thus, an electron in a negative energy state would repel an ordinary electron but would itself be accelerated towards the electron in the positive energy state. The principle of action and reaction would not be valid.

In the classical theory, however, no difficulty arises, because one can define the energy to be the positive square root of (1); and then it does not change with time.

This is not possible in the quantum theory. In the following sections we shall see that an external field (if it varies sufficiently rapidly) can cause transitions from a state of positive energy to a state of negative energy. The latter, therefore, cannot be excluded from the theory and, if the theory is not entirely wrong, must have some physical meaning.

In the applications of Dirac's wave equation, for instance, to the theory of the Compton effect, we have made use of the negative energy states as intermediate states. The results of the theory (Klein-Nishina formula) were found to be in very good agreement with the experiment up to energies of at least $10mc^2$. If we had only taken intermediate states with positive energy, we should have obtained a formula for the Compton scattering deviating largely from the Klein-Nishina formula, and the agreement with the experi-

The provisional character of these assumptions is evident. Before discussing the difficulties and some general features of this scheme, we shall show that it includes a theory of the positive electrons:

We consider what happens when one of the electrons with energy $E = -|E|$ and momentum $\mathbf{p}$ is removed under the action of an external field. By the second assumption the whole system then has a charge, energy, and momentum different from zero:

$$E_+ = -E = |E|, \qquad e_+ = -e, \qquad \mathbf{p}_+ = -\mathbf{p}, \tag{2}$$

where e represents the charge of an ordinary electron. The *hole* in the distribution of the negative energy electrons has therefore a *positive charge*, *positive energy*, and a momentum opposite to that corresponding to the negative energy state. It therefore behaves like an ordinary particle with electronic mass but with a positive charge. Energy and momentum are now connected by the equation

$$E_+ = +\sqrt{(p_+^2 + \mu^2)} \tag{3}$$

with the positive sign of the square root.

Thus, in the above picture, the positive electrons are represented as holes in the distribution of electrons filling up the negative energy states.

The creation and annihilation of a 'pair' of a positive and negative electron by an external field has to be interpreted in the following way.

Starting from a state where no particle is present, an external field acting on the electrons in the negative energy states may cause a transition of one of these electrons with energy E and momentum $\mathbf{p}$ to a state with positive energy E' and momentum $\mathbf{p}'$. According to (2) we have after the transition a pair of (positive and negative) electrons present with energies and momenta:

$$\begin{aligned} \mathbf{p}_+ &= -\mathbf{p}, & E_+ &= -E = |E|, \\ \mathbf{p}_- &= \mathbf{p}', & E_- &= E'. \end{aligned} \tag{4}$$

Since the minimum value of E' is equal to $+mc^2$ and the maximum value of E to $-mc^2$, the energy required to cause this transition must be larger than $2mc^2$:

$$E' - E = E_+ + E_- \geqslant 2mc^2. \tag{5}$$

On the other hand, if a negative and a positive electron are present initially, the negative electron can jump into the hole representing the positive electron and the pair is annihilated. By this process the

electron with the momentum $\mathbf{p}_0 = 0$ jumps into the 'hole' with the momentum $\mathbf{p}_0^*$, emitting the light quantum $\mathbf{k}$,

$$\mathbf{k} = \mathbf{k}_0 + \mathbf{p}_0 - \mathbf{p} = -\mathbf{p}_0^* \quad \text{(see Fig. 16).}$$

In the process the original electron has changed places with the electron in the negative energy state $\mathbf{p}_0^*$.

In a similar way we obtain a second intermediate state:

II. $\mathbf{k}$ is emitted by an electron in a negative energy state with momentum $\mathbf{p}_0^{**} = \mathbf{p} + \mathbf{k} = \mathbf{p}'^*$ going over into the state $\mathbf{p}$ with positive energy (the energy is, of course, not conserved for the intermediate states). In the transition to the final state the first

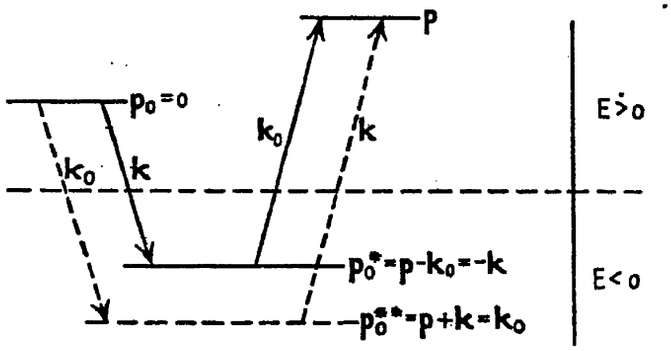

FIG. 16. The two intermediate states (drawn full and dotted) corresponding to the creation of a pair in the theory of the Compton effect according to the hole theory.

electron jumps into the hole, absorbing the light quantum

$$\mathbf{k}_0 = \mathbf{p} + \mathbf{k} = \mathbf{p}_0^{**} \quad \text{(see Fig. 16, dotted portion).}$$

Thus the transitions to the negative energy state used in § **16** are replaced by an *intermediate creation of a pair*. The matrix element arising from these intermediate states can easily be seen to be identical with those obtained from the original assumption (7) of intermediate transitions to negative energy levels.

The same is true for other processes in which transitions to negative energy states occur as intermediate states (§§ **16, 17, 18**). The hole theory, therefore, does not necessitate any alterations in the calculations of Chapter III.†

3. *Difficulties and general features of the hole theory.* While there can be no doubt that the general principles of this 'hole theory' are correct (creation of pairs observed directly and used as intermediate states in the theory of the Compton effect), the present form of the

† This, however, is not true for the transverse self-energy § 18 subsection 2. Compare the remark at the end of that subsection.

pair formation is essential. The 'switching on' of a static field represents a canonical transformation, and we see that in these transformations the total number of particles (positive and negative electrons) does not remain constant. It is therefore impossible to formulate exactly the problem fundamental for the present quantum mechanics and transformation theory of a *single* electron in an *external field*. In the exact theory of the future the total number of particles in a system will not be constant, the only constant being the total charge e, since e, of course, is invariant under pair production and annihilation.

At present we do not know whether the hole theory leads to correct results for positive electrons moving in a *strong field*. In all the applications of the following paragraphs, the positive electron can be considered as free or almost free. The interaction with other particles is taken into account only in the first approximation. Only in this case and if the probability for the creation or annihilation of positive electrons is small (compare below, under (*b*), (*c*)) can we expect the hole theory to give correct results.†

(*b*) Also the idea of an *electromagnetic field in vacuo* has to be *abandoned*. Even if no particles are present an electromagnetic field can give rise to the creation of pairs, Since, however, for this purpose a minimum energy of $2mc^2$ is required, pairs can only be created if, in the Fourier expansion of the field, frequencies higher then $2mc^2/\hbar$ or wave-lengths smaller than $\lambda_0/2$ occur. If this is the case, a 'pure field' *in vacuo* no longer exists. In the future theory the electromagnetic field and the 'field' representing the positive and negative electrons will be intimately connected, neither of them having a physical meaning independently from the other.

The Compton wave-length λ_0 represents a certain lower limit to the dimensions of a region in which the idea of pure electrodynamics *in vacuo*, as described by the quantized Maxwell equations (§ 7), has a precise physical meaning.

† It is because of the above difficulty that Dirac's wave equation has no stationary solution for an electron moving in the Coulomb field of a nucleus with a charge $Z > 137$. The radius of the K-shell would then be of the order $\hbar/mc$ and the potential energy of the order mc^2. In such a field positive and negative energy levels become mixed and it is not possible to define unambiguously the state of a single electron.

This break-down of Dirac's wave equation does not, however, seem to give rise to an appreciable error in the relativistic wave functions of a K-electron even for elements as heavy as lead ($Z = 82$). This follows, for instance, from the relativistic calculations of the photoelectric effect for heavy elements which have been found to agree very well with the experiments (§ 13 subsection 3).

In the transition to the final state $\mathbf{k}_1'$, $\mathbf{k}_2'$ the electron jumps into the hole (pair annihilated) emitting the quantum $\mathbf{k}_2'$:

$$\mathbf{p}_-'' = \mathbf{p}_0^* + \mathbf{k}_2'. \tag{8 d}$$

This scattering of light by light can even occur if the wave-length of the primary quanta k_1 and k_2 is larger than λ_0. The probability for the scattering is then, however, very small.

Processes of this sort can never be described by the present electrodynamics. Formally they can be obtained only from a *non-linear electro-dynamics for the vacuum*, i.e. from a theory in which the *principle of superposition of two fields is not valid.*

It is, however, doubtful whether the present hole theory can be applied to these processes.

From equations (8 a) to (8 d) it can be seen that the number of intermediate states is infinite because an infinite number of electrons in the negative energy states is available ($\mathbf{p}_0^*$ can have any value). For some simple examples it can be seen that the summation over this infinite number of intermediate states does not always converge. For instance, a light quantum has an *infinite self-energy* which is due to the fact that an infinite number of pairs can be created in intermediate states. (The scattering of light by light, however, converges.)

From the formal point of view the situation is very similar to that pointed out in § **18** for the higher approximations of the radiation processes. In that case the divergence of the summations was due to the fact that an infinite number of different light quanta could be emitted and absorbed in intermediate states.

From these difficulties we may conclude that the *hole theory in its present stage can in general only be applied in the first approximation*, in which a *finite number of pairs is created* (even in intermediate states) and in which the probability for creation is small. This will be the case in all applications of the following paragraphs.

We shall return to the question of the subtraction of the divergencies in § **25**.

20. Creation of positive electrons

According to the ideas of the preceding paragraph the creation of a pair of positive and negative electrons must be interpreted as a transition of an ordinary electron from a state of negative energy to a state of positive energy. The energy necessary to create a pair of *free* electrons with energies E_+ and E_- is equal to $E_+ + E_-$ and is

(i.e. by v_0). Thus formula (13) § 17 for the differential cross-section has to be multiplied by

$$\frac{\rho_{E_+}\rho_{E_-}dE_+}{\rho_E\rho_k dk}\frac{p_0}{E_0}=\frac{p_-^2\,dE_+}{k^2\,dk}. \tag{4}$$

Since $\mathbf{p}_0=\mathbf{p}_-$, $\mathbf{p}=-\mathbf{p}_+$, the angles θ, θ_0, ϕ, denoting the direction of the electron in the initial and final state, are connected with the angles θ_+, θ_-, ϕ_+, denoting the direction of the positive and negative electron, by:

$$\theta_+=\pi-\theta,\qquad \theta_-=\theta_0,\qquad \phi_+=\pi+\phi \tag{5}$$

($\theta_+=\angle(\mathbf{kp}_+)$, $\phi_+=\angle$ between $(\mathbf{kp}_+)$-plane and $(\mathbf{kp}_-)$-plane).

Putting then

$$E_0=E_-,\qquad E=-E_+,\qquad p_0=p_-,\qquad p=p_+ \tag{6}$$

and inserting (4), (5), (6) in the formula (13) § 17 we obtain the differential cross-section for the creation of a pair $\mathbf{p}_+$, $\mathbf{p}_-$:

$$\begin{aligned}d\phi=-\frac{Z^2}{137}\frac{e^4}{2\pi}\frac{p_+p_-dE_+}{k^3}\frac{\sin\theta_+\sin\theta_-d\theta_+d\theta_-d\phi_+}{q^4}\times\\ \times\left\{\frac{p_+^2\sin^2\theta_+}{(E_+-p_+\cos\theta_+)^2}(4E_-^2-q^2)+\frac{p_-^2\sin^2\theta_-}{(E_--p_-\cos\theta_-)^2}(4E_+^2-q^2)+\right.\\ +\frac{2p_+p_-\sin\theta_+\sin\theta_-\cos\phi_+}{(E_--p_-\cos\theta_-)(E_+-p_+\cos\theta_+)}(4E_+E_-+q^2-2k^2)-\\ \left.-2k^2\frac{p_+^2\sin^2\theta_++p_-^2\sin^2\theta_-}{(E_--p_-\cos\theta_-)(E_+-p_+\cos\theta_+)}\right\},\end{aligned} \tag{7}$$

$$q^2=(\mathbf{k}-\mathbf{p}_+-\mathbf{p}_-)^2. \tag{7a}$$

The integration over the angles is also just the same as for Bremsstrahlung. Formula (16) § 17 therefore has only to be multiplied by $p_-^2\,dE_+/k^2\,dk$ and E to put equal to minus the energy of the positive electron. The cross-section for the creation of a positive electron with energy E_+ and a negative one with energy E_- then becomes:†

$$\begin{aligned}\phi_{E_+}dE_+=\phi\frac{p_+p_-}{k^3}dE_+\left\{-\frac{4}{3}-2E_+E_-\frac{p_+^2+p_-^2}{p_+^2p_-^2}+\right.\\ +\mu^2\left(\frac{E_+\epsilon_-}{p_-^3}+\frac{\epsilon_+E_-}{p_+^3}-\frac{\epsilon_+\epsilon_-}{p_+p_-}\right)+L\left[\frac{k^2}{p_+^3p_-^3}(E_+^2E_-^2+p_+^2p_-^2)-\right.\\ \left.\left.-\frac{8}{3}\frac{E_+E_-}{p_+p_-}-\frac{\mu^2k}{2p_+p_-}\left(\frac{E_+E_--p_-^2}{p_-^3}\epsilon_-+\frac{E_+E_--p_+^2}{p_+^3}\epsilon_++\frac{2kE_+E_-}{p_+^2p_-^2}\right)\right]\right\}.\end{aligned} \tag{8}$$

† A provisional estimate of the probability of this process was first given by J. R. Oppenheimer and M. S. Plesset, *Phys. Rev.* **44** (1933), 53. Formula (8) was

This factor (12) destroys the symmetry in E_+ and E_-; this is because the positive electron is repelled and the negative electron attracted by the nucleus. According to (12) the probability for the creation of a pair is very small for small p_+ and large for small p_-. A similar correction must also be made in the relativistic case v_+, $v_- \sim c$ for *heavy elements*. It seems, however, that even for lead ($Z = 82$) this correction is not very large (compare subsection 2).

The screening need only be taken into account if both electrons have energies large compared with mc^2. In this case formula (9) is correct if the condition (11) is satisfied. In the opposite case of '*complete screening*', i.e. if $2E_+E_-/k\mu \gg 137Z^{-\frac{1}{3}}$, we obtain a formula corresponding to (26) § **17**:†

$$\phi_{E_+}\,dE_+ = 4\phi\,dE_+\left[\frac{E_+^2+E_-^2+\frac{2}{3}E_+E_-}{k^3}\log(183Z^{-\frac{1}{3}})-\frac{1}{9}\,\frac{E_+E_-}{k^3}\right]. \qquad \text{E.R.} \quad (13)$$

In our calculations we have not taken into account an *interaction* between the *positive* and *negative electron*. In the present theory of the positive electron this interaction is not included in a satisfactory way. Fortunately, it can be seen that to our approximation this interaction does not affect the calculation. This is because the matrix element $(V_{+-})_{p_+p_-}$ corresponding to the creation is probably the matrix element of a Coulomb interaction belonging to a transition from a positive energy state $\mathbf{p}_-$ to a state with negative energy and momentum $-\mathbf{p}_+$, viz.:

$$(V_{+-})_{p_+p_-} = \int \frac{d\tau\, e^{i(\mathbf{p}_-\mathbf{r}_-)/\hbar c}e^{i(\mathbf{p}_+\mathbf{r}_+)/\hbar c}}{|\mathbf{r}_+-\mathbf{r}_-|}. \qquad (14)$$

This matrix element vanishes except when momentum is conserved, i.e. $\mathbf{p}_++\mathbf{p}_- = 0$. It follows that V_{+-} will not contribute anything to the matrix elements occurring in § **17** (8). Thus V_{+-} can be neglected in our approximation.

2. *Discussion. Total number of pairs.* Discussing first the angular distribution of the created electrons, we see from (7) that for large energies, $k, E_+, E_- \gg \mu$, both electrons are emitted in the forward direction (as is the case for all processes at very high energies). For smaller energies the concentration in the forward direction is less marked. The solid angle θ in which the electrons are emitted is of the order $\theta \sim mc^2/k$.

† H. Bethe, *Proc. Camb. Phil. Soc.* xxx. 524, and Bethe and Heitler, loc. cit.

obtains a very small and the other one a very large energy. Finally, the distribution tends to an asymptotic curve (∞) represented by formula (13).

The symmetry in E_+ and E_- is due to the application of Born's approximation. In an exact calculation the maximum of the distribution would be displaced to the right-hand side. This displacement is greatest for high nuclear charge Z and small k.

The total number of pairs created is found by integration of (8)–(13) over all possible energies of the positive electron; it is thus equal to the area covered by one of the curves of Fig. 17. Analytic integration is only possible in the extreme relativistic case if the screening is either negligible or complete. From (9) and (13) we obtain

$$\phi_{\text{pair}} = \bar{\phi}\left(\frac{28}{9}\log\frac{2k}{\mu} - \frac{218}{27}\right); \qquad \text{E.R.} \quad (15)$$

$$\phi_{\text{pair}} = \bar{\phi}\left(\frac{28}{9}\log(183Z^{-\frac{1}{3}}) - \frac{2}{27}\right). \qquad \text{E.R.} \quad (16)$$

For smaller energies and for energies where the screening is not complete the integration was carried out numerically (loc. cit.).

The results are shown in Fig. 18, where we have plotted in a logarithmic scale the total cross-section ϕ_{pair}, in units $\bar{\phi}$, as a function of $\hbar\nu$. Fig. 18 shows that the probability of the pair formation increases rapidly with $\hbar\nu$ until it reaches a constant value at very high energies. It is proportional to Z^2 (because $\bar{\phi} \sim Z^2$).† The values of $\phi_{\text{pair}}/\bar{\phi}$ are given in the following table (compare also Appendix II):

TABLE V

Cross-section for the creation of pairs by γ-rays

$\hbar\nu$	3	4	5	6	10	20	50	100	$\infty\, mc^2$
$\phi_{\text{pair}}/\bar{\phi}$	0·085	0·32	0·61	0·89	1·94	3·75	6·4	7·9 (Pb)	11·5 (Pb)

We can compare the probability for the creation of a pair by a γ-quantum k in the field of a nucleus with charge Z with the probability, for the same γ-quantum, of Compton scattering by the extra nuclear electrons. The latter is given by the Klein-Nishina formula § **16** eq. (53) multiplied by the number of electrons Z. We have therefore also plotted in Fig. 18 the cross-section for the Compton scattering in the same units $\bar{\phi}$. In these units we obtain,

† For very high energies $\phi_{\text{pair}}/\bar{\phi}$ depends also on Z, but only to a small extent.

positive electrons; the number of photoelectrons is not very large for the energies in question.

Thus, by counting the numbers n_+, n_- of the positive and negative electrons we obtain immediately the ratio of the cross-sections

$$\phi_{\text{pair}}/(\phi_{\text{photoelectr.}}+\phi_{\text{Compton}}) = n_+/(n_- - n_+).$$

$\phi_{\text{photoelectr.}}+\phi_{\text{Compton}}$ can be taken from the calculations §§ **13, 16** which have been found to agree very well with the experiments. The result of the measurements reduced to an infinitely thin sheet for the ThC″ γ-radiation ($k = 5{\cdot}2mc^2$) and Pb is shown in Fig. 18. The theoretical value of the ratio $n_+/(n_- - n_+)$ for this energy is 0·20, the experimental value 0·22.

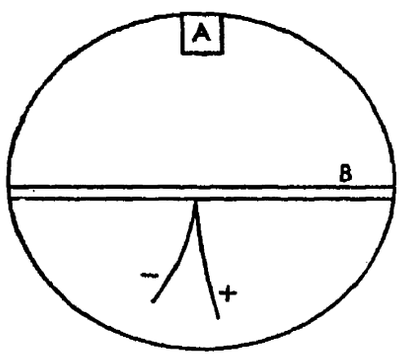

FIG. 19. Scheme of experimental arrangement for the observation of the creation of pairs by γ-rays in a Wilson chamber. A = source of γ-rays. B = thin lead target. A strong magnetic field is switched on deflecting positive electrons travelling downwards to the right-hand side. +, − tracks of the pair.

As is seen from these data, the *theory of the pair formation agrees very well with the experimental results.* The agreement is even better than was to be expected, since we should not expect Born's approximation to give very accurate results for elements as heavy as lead (compare Appendix II). The proportionality with Z^2 has been tested experimentally in a similar way by Benedetti† and found to be satisfied with great accuracy.

The theory of pair formation can also be tested in a more indirect way. The creation of pairs gives rise to an additional absorption of γ-rays. The absorption coefficient per cm. due to pair formation is given by

$$\tau_{\text{pair}} = N\phi_{\text{pair}}, \tag{17}$$

where N represents the number of atoms per cm.^{3} Thus, by measuring the absorption coefficient of γ-rays we also obtain evidence for the creation of pairs. We shall, however, leave this question for a special

† S. de Benedetti, *C.R.* **200** (1935), 1389.

Experimentally, this process seems to have been observed by Skobelzyn and Stepanowa.† Eq. (19) is also valid for a proton with energy $E_0 \gg Mc^2$, if μ is replaced by Mc^2. In the calculation of the energy loss by fast particles in passing through matter (§ 23) the creation of pairs can be neglected.

(3) A special discussion is required for the case of pair formation by a direct *collision between two electrons*.‡ The conservation laws show that this happens only if the colliding electron has an energy larger than $7mc^2$ (if the other electron is initially at rest). This case is essentially different from (2) because the electron which was initially at rest obtains a large velocity during the process. Thus the *retardation* forces between all three electrons will play an important part.

(4) *In vacuo* pairs can be created by *two light quanta* $\hbar\nu_1$ and $\hbar\nu_2$ with an energy $\hbar\nu_1+\hbar\nu_2 > 2mc^2$. This is the inverse process to the ordinary two-quanta annihilation which we shall consider in § 21.‖ The probability of the pair formation can be deduced immediately from formula (11) § 21, and for ordinary radiation densities is extremely small. For densities, however, which are estimated to occur in the interior of the stars, the pair formation by light may become important. In thermodynamic equilibrium, an appreciable number of pairs is present at temperatures of $kT \sim mc^2$ (i.e. $T \sim 5.10^9$ °C.). For higher temperatures the density of pairs becomes equal to the density of the light quanta themselves.

(5) Finally, we mention the fact that a γ-ray emitted by a nucleus can create a pair in the field of the same nucleus. This process is a kind of *internal conversion*, with the creation of a pair instead of ejection of a K-electron. For γ-rays of energy about $5mc^2$ the coefficient of this internal conversion is of the order 10^{-4}–10^{-3} pairs per γ-quantum.††

21. The annihilation of positive electrons

The inverse process to the creation of pairs is the *annihilation* of a positive and a negative electron. According to the hole theory (§ 19), this has to be understood as a transition of an ordinary electron from

† D. Skobelzyn and E. Stepanowa, *Journ. d. Phys.* **6** (1935), 1.

‡ F. Perrin, *C.R.* **197** (1934), 1100, 1302.

‖ G. Breit and J. A. Wheeler, *Phys. Rev.* **46** (1934), 1087; W. Heitler, *Proc. Camb. Phil. Soc.* **31** (1935), 242.

†† J. R. Oppenheimer and L. Nedelski, *Phys. Rev.* **44** (1933), 948; M. E. Rose and G. E. Uhlenbeck, ibid. **48** (1935) 211; J. C. Jaeger and H. R. Hulme, *Proc. Roy. Soc.* **148** (1935), 708.

instance, from the initial state to the first intermediate is given by (§ **10** eq. (16))

$$H_{AI} = -e\sqrt{\left(\frac{2\pi\hbar^2c^2}{k}\right)}(u_0^*\,\alpha_1\,u'), \tag{4}$$

where u_0, u' denote the Dirac amplitudes of the electron in the states $\mathbf{p}_0$, $\mathbf{p}'$, and α_1, α_2 the components of the matrix vector $\boldsymbol{\alpha}$ in the directions of polarization of the two light quanta.

To the approximation used in calculating the transition probability, the interaction between the positive and negative electron vanishes (see § **20** eq. (14)).

Thus the matrix element responsible for the annihilation process is according to (4), given by

$$\begin{aligned} H &= \sum\left(\frac{H_{AI}H_{IF}}{E_A-E_I}+\frac{H_{AII}H_{IIF}}{E_A-E_{II}}\right) \\ &= \frac{e^2 2\pi\hbar^2c^2}{k}\sum\left(\frac{(u_0^*\alpha_1 u')(u'^*\alpha_2 u)}{E'}+\frac{(u_0^*\alpha_2 u'')(u''^*\alpha_1 u)}{E''}\right), \end{aligned} \tag{5}$$

where the summation $\sum$ is over all spin directions and energy signs in the intermediate states.

To obtain the transition probability we must remember that the final state is determined by the direction $\mathbf{k}_1$ of one light quantum only, the direction of $\mathbf{k}_2$ being, by (2), opposite to that of $\mathbf{k}_1$. Therefore we have to multiply $|H|^2$ by the density function $d\Omega\, k_1^2\, dk_1/(2\pi\hbar c)^3$, referring to the first light quantum only. Dividing by the velocity of the incident positive electron p_+c/E_+ we obtain the differential cross-section

$$\begin{aligned} d\phi &= \frac{2\pi}{\hbar c}\frac{E_+}{p_+}\frac{d\Omega\, k_1^2}{(2\pi\hbar c)^3}|H|^2 \\ &= \frac{2\pi}{\hbar c}\frac{E_+}{p_+}\frac{d\Omega\, k_1^2}{(2\pi\hbar c)^3}\frac{e^4(2\pi\hbar^2c^2)^2}{k_1^2}\left|\sum\left(\frac{(u_0^*\alpha_1 u')(u'^*\alpha_2 u)}{E'}+\frac{(u_0^*\alpha_2 u'')(u''^*\alpha_1 u)}{E''}\right)\right|^2. \end{aligned} \tag{6}$$

The summation $\sum$ can be carried out by the method developed in the theory of the Compton effect (§ **16** subsection **3**). We shall give the results only. Summing the cross-section over the directions of polarization of the two light quanta and averaging over the spin directions of the positive and negative electron, we obtain

$$d\phi = \frac{e^4}{4p_0E_0}\left(\frac{E_0^2+p_0^2+p_0^2\sin^2\theta}{E_0^2-p_0^2\cos^2\theta}-\frac{2p_0^4\sin^4\theta}{(E_0^2-p_0^2\cos^2\theta)^2}\right)\sin\theta\, d\theta d\phi, \tag{7}$$

cross-section for annihilation diverges. This, however, does not mean that the probability of annhilation becomes infinite. The rate of annihilation in a substance with N atoms per unit volume for this case is given by

$$R = ZN\phi v_+ = NZ\pi r_0^2 c \text{ (sec.}^{-1}\text{)}. \qquad \text{N.R.} \quad (12)$$

For lead, for instance, we obtain $R = 2 \times 10^{10}$ sec.$^{-1}$. The lifetime of a very slow positive electron in lead is therefore of the order 10^{-10} sec.

For high energies the cross-section decreases. For very high energies we obtain

$$\phi = \pi r_0^2 \frac{\mu}{E_+}\left(\log \frac{2E_+}{\mu} - 1\right). \qquad \text{E.R.} \quad (13)$$

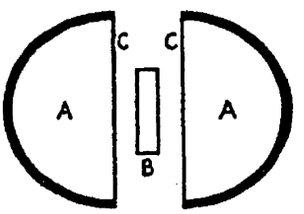

FIG. 20. Scheme of an experimental arrangement for the observation of the annihilation radiation. A = two Geiger-Müller counters. B = source of positive electrons surrounded by a thin sheet of metal. C = thin wall of the counters. (Klemperer, loc. cit.)

In the Lorentz system in which the negative electron is at rest, the two annihilation quanta do not in general have the same frequency. For *high energies* of the positive electron, we see from the angular distribution function (7) that, in the initial Lorentz system, the two quanta are mainly emitted in the forward and backward directions. After the Lorentz transformation the *quantum* emitted in the *forward direction* takes up nearly all the *energy of the positive electron*, whereas the second quantum has only an energy of the order mc^2. If, however, the kinetic energy of the positive electron is small compared with mc^2, the two quanta have an *energy of* mc^2 *each* and are emitted in *opposite directions*.

From (11) we can deduce the probability of annihilation of a positive electron in passing through a finite sheet of matter. This will be done in § 23.

2. *Experimental evidence.*† The above theory of annihilation can be

† O. Klemperer, *Proc. Camb. Phil. Soc.* **30** (1934), 347. The source of positive electrons was a target of graphite bombarded by protons. A radioactive element N_{13} is thus produced, which emits positive electrons. Compare also: J. Thibaud, *C. R.*, **197** (1933), 1629; F. Joliot, ibid. **198** (1934), 81; H. R. Crane and C. C. Lauritsen, *Phys. Rev.* **45** (1934), 430.

process, and amounts in heavy elements (in which it is largest) to less than 20 per cent. of the two-quanta annihilation.

We shall calculate the probability only in the most simple case where the negative electron is bound in the K-shell of an atom, and where the kinetic energy of the positive electron is small compared with mc^2 but large compared with the ionization energy of the K-shell. The latter assumption makes possible the use of Born's approximation. The calculation is then very similar to that for the photoelectric absorption of a light quantum by the K-electron (see § 13 subsection 1, where the same assumptions are made).

The process in question consists of a transition of an electron from the K-state to a state of negative energy with momentum $\mathbf{p} = -\mathbf{p}_+$, emitting a light quantum

$$k = \sqrt{(\mu^2+p_+^2)}+\mu-\frac{\alpha^2}{2\mu}, \tag{14}$$

where $\alpha^2/2\mu$ represents the binding energy of the K-electron. According to the assumptions made above we can assume the positive electron to be free (Born's approximation). For its wave function we may take the non-relativistic approximation.†

The transition to the non-relativistic case requires, however, some care. We start from the exact matrix element responsible for the transition

$$H = -e\sqrt{\left(\frac{2\pi\hbar^2c^2}{k}\right)}(\psi_K^*\alpha_e\psi_p), \tag{15}$$

where α_e represents the component of the matrix vector $\boldsymbol{\alpha}$ in the direction of polarization of the light quantum, ψ_K, ψ_p the exact Dirac wave function of the K-electron and the electron in the negative energy state.

The expression $\alpha_e\psi_p$, which has four components, can be written in the non-relativistic approximation as a scalar quantity. For this purpose we write Dirac's wave equation in the form

$$(\boldsymbol{\alpha}\mathbf{p})\psi_p = (E-\beta\mu)\psi_p, \tag{16}$$

where

$$E = \pm\left(\mu+\frac{p^2}{2\mu}\right), \tag{17}$$

the two signs referring to positive and negative energy states respectively. Inserting (17) in (16) and multiplying by α_x, putting it both

† The word 'non-relativistic' is not, of course, to be understood literally. The whole effect is essentially a relativistic one. It means only that $\mathbf{p}_+ \ll \mu$.

amounts to about $25r_0^2$. For the same energy the cross-section for the two-quanta annihilation is sixteen times larger. In contrast to the two-quanta annihilation, ϕ_K decreases for small energies. For relativistic energies the exact formula is

$$\phi_K = 4\pi r_0^2 \frac{Z^5}{137^4} \frac{1}{(\gamma+1)^2(\gamma^2-1)^{\frac{1}{2}}} \times \times \left[\gamma^2 + \tfrac{2}{3}\gamma + \tfrac{4}{3} - \frac{\gamma+2}{(\gamma^2-1)^{\frac{1}{2}}} \log\{\gamma+(\gamma^2-1)^{\frac{1}{2}}\}\right] \quad (20)$$

$$(\gamma = E_+/\mu).$$

The ratio of one-quantum annihilation to two-quanta annihilation is largest for about $E_+ \sim 10mc^2$, where it amounts to ~ 20 per cent. (for lead).† These values are obtained by using Born's approximation. It is likely that the correct values of ϕ_K for lead are still considerably smaller as is the case for the photoelectric effect. (Compare Tables I and II, p. 126.)

Positive electrons can also be annihilated without emission of radiation, the energy being given off to another electron. This process may occur if a positive electron collides with two electrons which are accidentally situated very near to each other,‡ or by a collision with the K-shell.‖ One electron is then annihilated whereas the other one is ejected with a kinetic energy of $\sim 2mc^2$. The probability for these processes is, however, rather small.

† For a more detailed discussion of the one-quantum annihilation see E. Fermi and G. E. Uhlenbeck, *Phys. Rev.* **44** (1933), 510; H. R. Hulme and H. J. Bhabha, *Proc. Roy. Soc.* **146** (1934), 723; Y. Nishina, S. Tomonaga, and H. Tamaki, *Scient. Pap. Inst. Phys. Chem. Research, Tokio*, **24** (1934), No. 18; H. Bethe, *Proc. Roy. Soc.* **150** (1935), 129; J. Solomon, *Journ. d. Phys.* **6** (1935), 114; J. C. Jaeger and H. R. Hulme, *Proc. Camb. Phil. Soc.* **32** (1936), 158.

‡ F. Perrin, *C.R.* **197** (1933), 1302.

‖ J. Brunings, *Physica*, **1** (1934), 996; H. S. W. Massey and E. H. S. Burhop, *Proc. Roy. Soc.* **167** (1938), 53.

If ϕ represents the cross-section of one atom for one of these three processes, the corresponding value of τ is given by

$$\tau = N\phi, \tag{3}$$

where N is the number of atoms per cm.^{3}

The numerical values of the cross-sections for the three processes, as calculated from the theory, are given in this book as follows:

(1) For the photoelectric effect of the K-shell† in Tables I, II, and Fig. 8, pp. 124, 126.

(2) For the Compton scattering in Table III and Fig. 13, pp. 158, 160.

The values refer to a single electron and have to be multiplied by Z to give the cross-section of an atom.

(3) For the creation of pairs in Table V and Fig. 18, p. 200, 201.

Since the quantities are expressed in units $\phi_0 Z^5/137^4$, $\phi_0 (= 8\pi r_0^2/3)$, $\bar{\phi} (= r_0^2 Z^2/137)$ respectively, the figures in the tables and graphs mentioned above have to be multiplied respectively by $N\phi_0 Z^5/137^4$, $NZ\phi_0$, $N\bar{\phi}$ to give the corresponding term in τ. The values of these quantities for several materials are given in the appendix.

With regard to the graphs Fig. 13 and Fig. 18, we see that the three processes are not equally important in different energy regions. For small quantum energies $\hbar\nu$ the photoelectric absorption gives the main contribution. For higher energies the Compton scattering becomes more important, and finally, for very high energies, the absorption is entirely due to pair production. The regions where these three effects give the largest contribution are roughly as follows:

	Photoelectric effect	*Compton effect*	*Pair formation*
Pb	$\hbar\nu/mc^2 < 1$	$\sim 1-10$	> 10
Al	$\hbar\nu/mc^2 < 0{\cdot}1$	$\sim 0{\cdot}1-30$	> 30

As a function of the atomic number Z, the absorption coefficient behaves most simply in the region where the Compton scattering gives the only contribution. τ in this region is simply proportional to the total number of electrons NZ per cm.^{3} or, since Z is roughly proportional to the atomic weight, proportional to the density ρ. Thus the *total intensity absorbed is proportional to the total mass per cm.*^{2} which the γ-ray has passed. We can then define a *mass absorption coefficient* τ/ρ (gm.$^{-1}$ cm.^{2}) which is almost constant for all elements at a

† The higher shells give only a small contribution amounting to about 20 per cent. of the effect of the K-shell. In the following we have multiplied ϕ_K by a factor 5/4. (Cf. § 13.)

materials in order to determine uniquely the frequency of a given radiation. A single measurement is only sufficient if the frequency is so small that pair production cannot occur. As an example, we have plotted in Fig. 21 the experimental values of τ for a certain γ-radiation obtained by bombarding fluorine with protons.† The value for Pb (0·49) allows two frequencies of about $\hbar\nu = 4{\cdot}5$ and 11 mc^2.

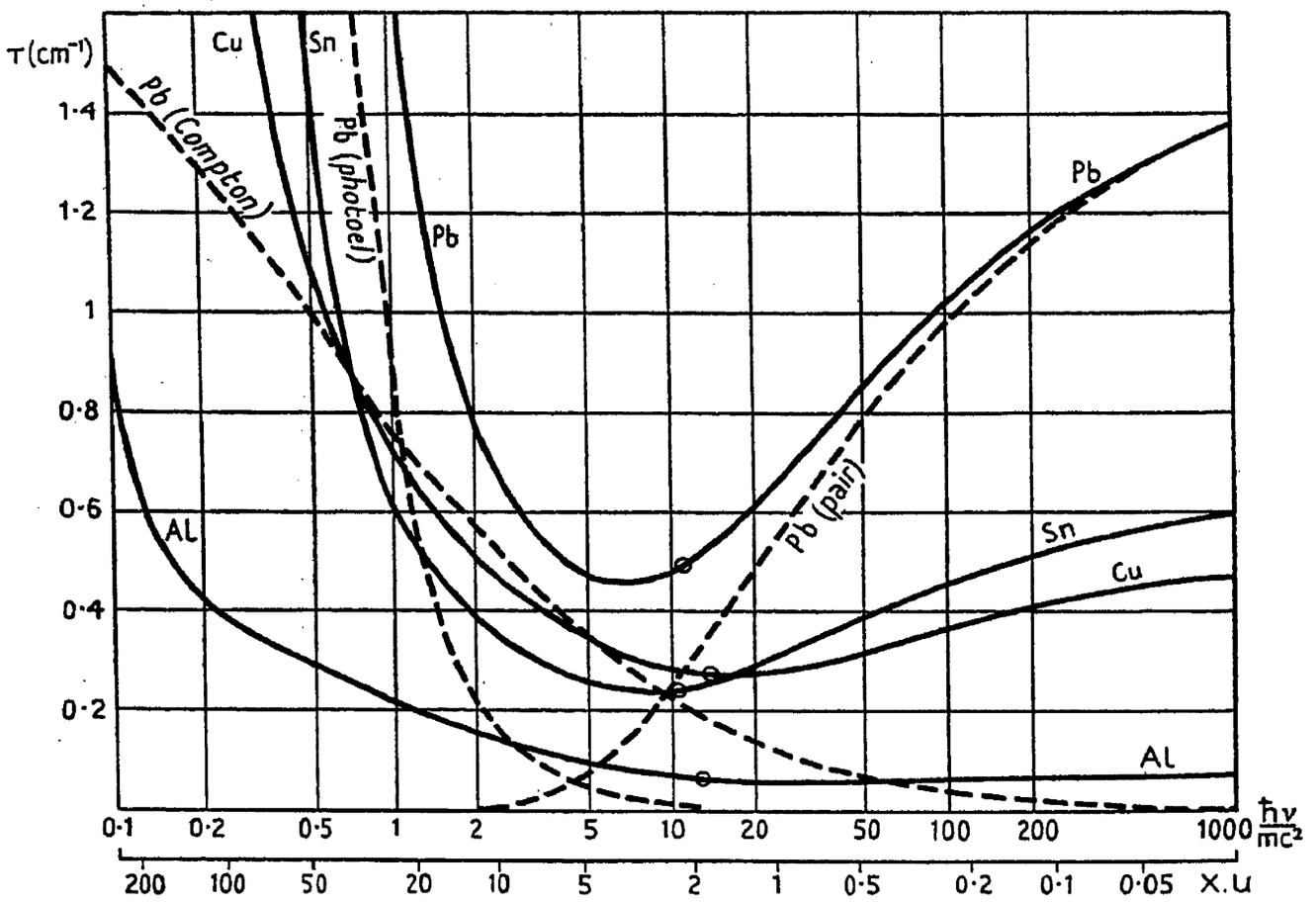

FIG. 21. Absorption coefficient τ for γ-rays in Pb, Sn, Cu, Al as a function of the frequency, in a logarithmic scale. The dotted curves show the three components of τ for lead. The circles refer to a measurement of McMillan (loc. cit.). They should lie on the same vertical line.

The values obtained for Sn, Cu, and Al show definitely that we are on the short wave-length side and that $11mc^2$ is the true value of $\hbar\nu$.

If the theory is correct, all experimental points should lie on the same vertical line. This is actually the case allowing for the fact that the accuracy of neither the measurements nor the calculations is higher than 10 per cent. and that $\hbar\nu$ is very sensitive to τ.

The above measurements can be regarded as a proof of the theory of pair production. From these results, and from the direct measure-

† McMillan, *Phys. Rev.* **46** (1934), 868. For other examples see H. R. Crane and C. C. Lauritsen, *Rep. Intern. Conference on Physics*, London, 1934.

the Bremsstrahlung gives rise to an extraordinarily high energy loss which considerably outweighs the energy loss due to inelastic collisions.

1. *Average energy loss by inelastic collisions.* The energy loss of particles passing through matter was first calculated by Bohr,† using the classical theory. In the quantum mechanics the problem has been investigated and solved by several authors‡ in a very satisfactory way. In this book, however, we must confine ourselves to a statement of the results of these beautiful but rather lengthy calculations. We take here the results obtained by Bloch who treated the problem using the Thomas-Fermi model for the atom.

The average energy lost by a particle of charge ze with the velocity $v = c\beta$ and energy E per cm. path in a substance with atomic number Z is given by the general formula:

$$\left(-\frac{dE}{dx}\right)_{\text{coll}} = NZ\phi_0\mu\tfrac{3}{4}z^2\frac{1}{\beta^2}\left[\log\frac{\mu\beta^2 W}{(1-\beta^2)I^2Z^2}+1-\beta^2+\Psi(0)-\mathrm{R}\Psi\left(i\frac{z}{137\beta}\right)\right]. \quad (1)$$

N is the number of atoms per cm.^{3}

ϕ_0 is our usual unit cross-section.

μ is the rest energy of the electron.

IZ is an average ionization energy of the atom. I has been determined from the experimental value of the energy loss of α-particles in gold:

$$I = 13{\cdot}5 \text{ electron volts.} \quad (2)$$

$\Psi(x)$ is the logarithmic derivative of the gamma function‖

$$\Psi(x) = \frac{d\log(x!)}{dx} = \frac{d\log\Gamma(x+1)}{dx}.$$

For large values of x, $\Psi(x)$ can be developed in a series

$$\Psi(x) = \log x + \frac{1}{2x} - \ldots. \quad (3)$$

R denotes the 'real part of'.

† N. Bohr, *Phil. Mag.* **25** (1913), 10; **30** (1915), 581.

‡ Cf. Mott and Massey, *Theory of Atomic Collisions*, Oxford, 1933, Chapter XI; Ch. Møller, *Ann. Phys.* **14** (1932), 531; H. Bethe, *Zs. f. Phys.* **76** (1932), 293; *Hdb. d. Physik*, XXIV (1), 521; E. J. Williams, *Proc. Roy. Soc.* **135** (1932), 108; F. Bloch, *Ann. Phys.* **16** (1933), 285; *Zs. f. Phys.* **81** (1933), 363.

‖ Ψ is tabulated in Jahnke-Emde, *Tables of Functions*, Leipzig, 1933:

$$\Psi(0) = -0{\cdot}577.$$

(5) becomes simply

$$\left(-\frac{dE}{dx}\right)_{\text{coll}} = NZ\phi_0\mu\frac{3}{4}\frac{\mu}{T}\left[\log\frac{T}{IZ}+\frac{1}{2}\right]. \qquad \text{N.R.} \quad (6)$$

In the extreme relativistic case $E \gg \mu$ we obtain

$$\left(-\frac{dE}{dx}\right)_{\text{coll}} = N\phi_0 Z\mu\tfrac{3}{4}\log\frac{E^3}{2\mu I^2 Z^2}. \qquad \text{E.R.} \quad (7)$$

From (6) and (7) we see that for small energies the *energy loss decreases rapidly with increasing energy*. It reaches a *minimum value* somewhere *at energies of the order* mc^2 and *then increases again*; the increase is, however, very slow.

According to (1), a *heavy particle* loses practically the same energy per cm. path (except for a factor z^2) as an electron of the *same velocity*. The comparison becomes even more exact if for the electron we disregard head-on collisions and insert for the maximum energy transferred a quantity of the order $W \sim mv^2$. (Compare footnote, p. 219.)

If, however, we consider an electron and a heavy particle with the *same energy* E, the heavy particle loses much more energy than the electron, at least for $E < Mc^2$. Introducing the kinetic energy of the heavy particle,

$$T = \frac{Mc^2}{\sqrt{(1-\beta^2)}} - Mc^2 \sim \frac{Mc^2}{2}\beta^2,$$

eq. (1) becomes (for $\beta \ll 1$ but $z/137\beta \ll 1$)

$$\left(-\frac{dE}{dx}\right)_{\text{coll}} = NZ\phi_0\tfrac{3}{4}z^2\frac{M}{m}\frac{\mu^2}{T}\left[\log\left(\frac{T}{IZ}\frac{m}{M}\sqrt{8}\right)+\tfrac{1}{2}\right]. \qquad \text{N.R.} \quad (8)$$

The energy loss due to inelastic collisions is very nearly proportional to the total number of electrons NZ per cm.^{3} or to the total mass per cm.^{2} traversed. It behaves in this respect like the absorption coefficient of γ-rays due to Compton scattering. For high energies (eq. (7)) this is also the case for the dependence on the primary energy. (We have then to compare τ with the energy loss divided by the primary energy E.) In fact, the collision of two electrons and of a light quantum with an electron are very similar processes for high energies.

The energy loss of a proton and an electron due to collisions in Pb is plotted in Fig. 22 as a function of the primary energy E.

2. *Total energy loss.* The second way in which an electron loses energy in passing through matter is by emission of radiation. The

At *very high energies the energy loss is almost entirely due to Bremsstrahlung*. The range of energies where this begins to be the case lies at about $20mc^2$ for Pb and $200mc^2$ for air and H_2O.

In Fig. 22 we have plotted the two parts and the total energy loss per cm. of an electron in lead. The shape of the curves is, as it is seen from the graph, very similar to the curves in Fig. 21 representing the absorption coefficient of γ-rays.

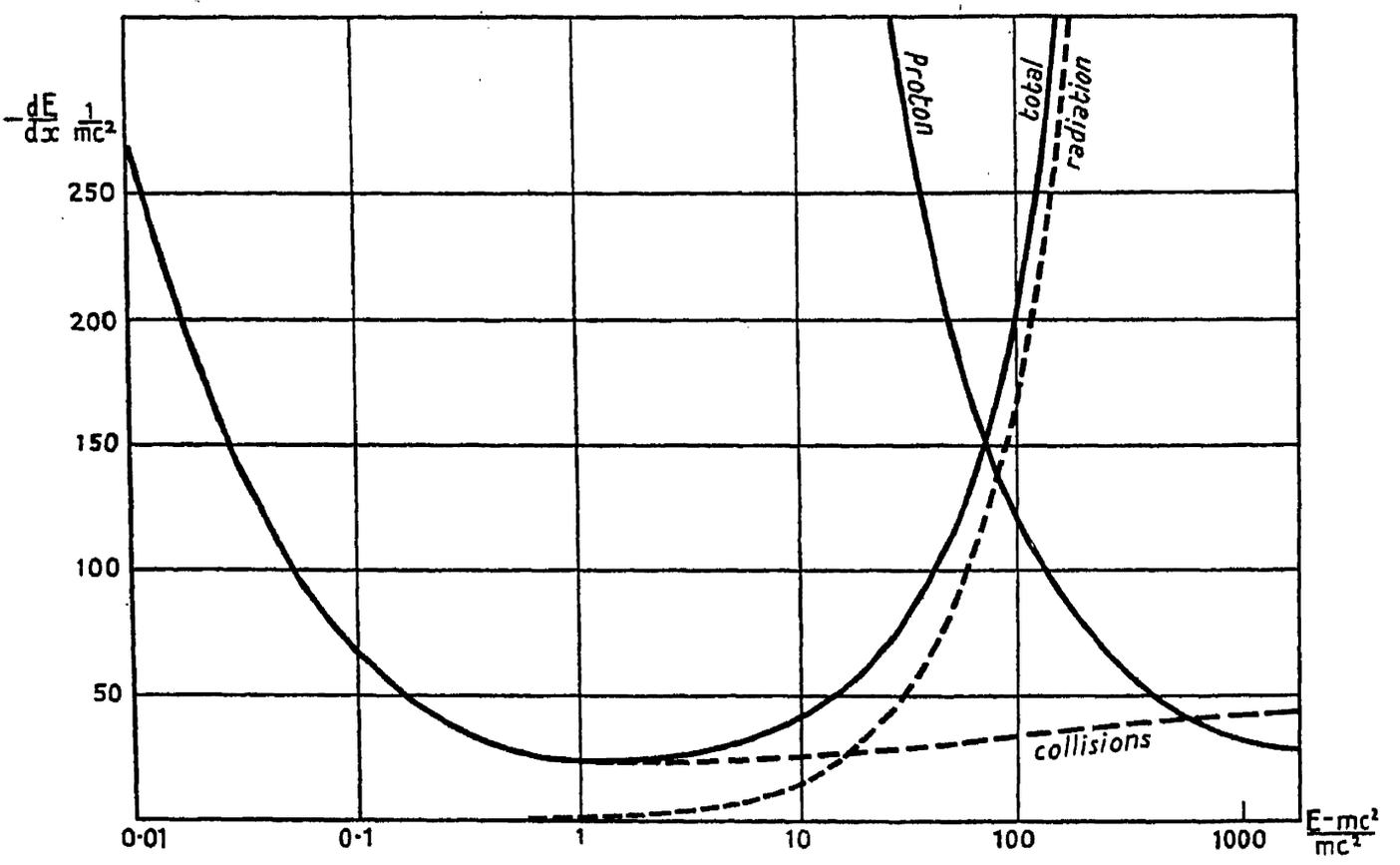

FIG. 22. Energy loss per cm. path in lead (units mc^2) as a function of the primary energy $E-mc^2$ in a logarithmic scale for an electron and a proton of the same energy. The dotted curves represent the two parts of the energy loss for the electron.

For a *heavy particle* the energy loss is entirely due to inelastic collisions. The radiative part will be smaller by a factor $(m/M)^2$ than for an electron of the same velocity. This can be seen even from the classical theory of emission of radiation (cf. § 17 subsection 5). The energy loss by radiation becomes comparable with that by inelastic collisions only at energies large compared with Mc^2.

For comparison we have also plotted in Fig. 22 the energy loss of a *proton* in Pb as given by equations (1) and (8). As we see from the graph, the *stopping power of Pb for energies larger than* $100mc^2$ *is smaller for protons than for electrons*. In air, this limit lies at about $200mc^2$.

from the actual amount of energy which is lost in a particular case by a particle traversing a sheet of matter of a certain thickness. If, however, the particle loses its energy in a very large number of collisions, transferring only a small fraction of its energy in each collision, the effect of straggling will be very small. This is always the case for heavy particles (cf. eq. (4 a)).

An electron may, however, lose a *large fraction of its energy* in a *single head-on collision, or by emission of a hard γ-ray*. Since the head-on collisions are comparatively rare, the straggling due to the in-

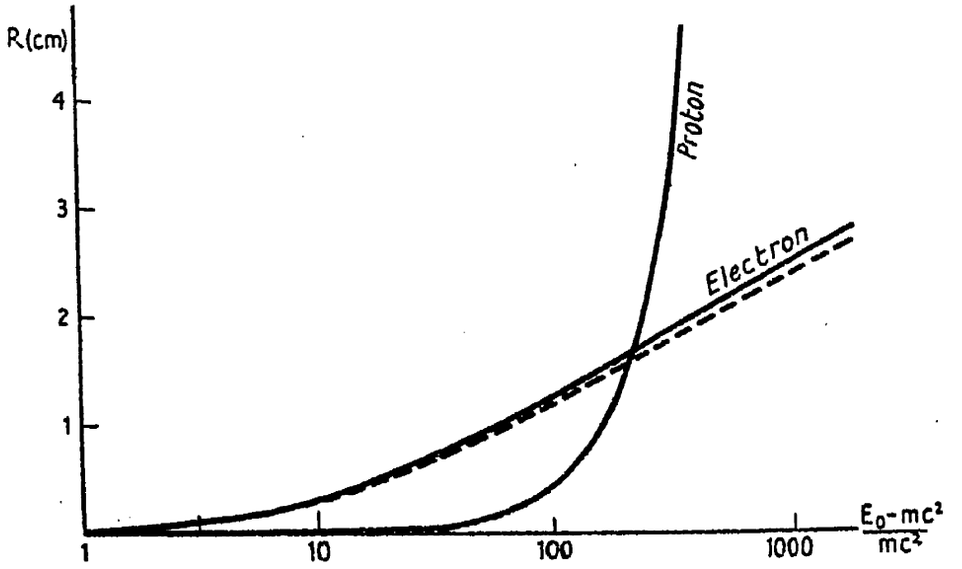

FIG. 23. Average range of a fast electron and proton of the same energy in lead. The dotted curve represents the average range of a positive electron, the difference being due to annihilation of the positive electron while in motion.

elastic collisions only will not be very large. We shall not consider it here.†

The *straggling* is, however, *characteristic of the energy loss by radiation*.‡ In § 17 (subsection 3, Fig. 14) we have seen that the probability that an electron loses a fraction $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ of its energy in a single-quantum process is almost the same in the three cases. Thus the energy loss by radiation is chiefly due to the emission of *large quanta*. After passing a certain sheet of matter, an electron will have emitted only a few quanta each of large energy. The secondary γ-radiation of an electron has an energy of the same order of magnitude as that of the primary electron. The straggling will therefore be very large.

To obtain a rough idea of the effect of this straggling we represent

† Cf., for instance, Bethe, *Hdb. d. Physik* (loc. cit.).
‡ H. Bethe and W. Heitler, loc. cit. p. 161.

Therefore, if (15) is valid for l_1 and l_2, it is also valid for l_1+l_2, and since we have shown that it is valid for an infinitely short path we have proved that (15) is valid for any length of the path.

The probability distribution given by formula (15) is plotted in Fig. 24 for several values of bl. The constant b can be taken from (12), (14), and from the appendix where the values of $N\bar{\phi}$ are given for several materials. For a sheet of lead of 1 cm. thickness bl amounts to 2·5. The graph shows that the probable energy losses are distributed over a broad range. For a sheet of Pb of 8 mm. thickness ($bl = 2$) the probability that the energy decreases by a factor between $e^{-0.5} = 0.6$ and $e^{-2} = 0.135$ is almost constant.

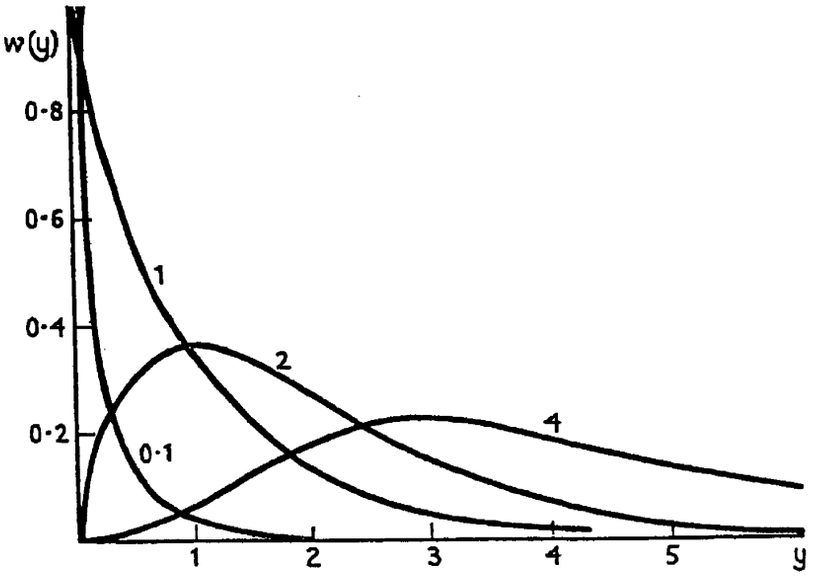

FIG. 24. Straggling. $w(y)$ represents the probability that the energy of an electron which has traversed a sheet l cm. thick has decreased by the factor e^{-y}. The numbers affixed to the curves represent the parameter bl, where b is a constant defined in equations (12) and (14). (For Pb $b = 2{\cdot}3$ cm.$^{-1}$)

The probability that an electron still has an energy larger than e^{-y_0} times the initial energy after traversing the sheet is given by the integral

$$W(y_0) = \int_0^{y_0} w(y)\,dy = \int_0^{y_0} \frac{e^{-y}y^{bl-1}}{\Gamma(bl)}\,dy = \frac{(bl-1, y_0)!}{\Gamma(bl)}, \qquad (17)$$

where $(bl-1, y_0)!$ represents the 'incomplete gamma function'.† For small values of y_0, that is, if we wish to know the probability that the electron loses only a very small fraction of its initial energy, we obtain

$$W(y_0) = \frac{(bl-1, y_0)!}{\Gamma(bl)} = \frac{y_0^{bl}}{\Gamma(bl+1)}. \qquad (17')$$

† Numerical values of this function are given in Jahnke-Emde, *Tables of Functions*, p. 96, and Pearson, *Tables of the Incomplete* Γ*-function*. $W(y_0)$ is plotted in H. Bethe and W. Heitler, loc. cit., p. 161.

made by Anderson and Neddermeyer,† and they have been able to verify the theoretical predictions for energies up to a few hundred mc^2. The most extensive and most accurate measurements are by Blackett and Wilson.‡

In Fig. 25 we represent Blackett's results. They were obtained with a 0·33 cm. Pb plate for energies up to $1{,}000mc^2$ and a 1 cm.

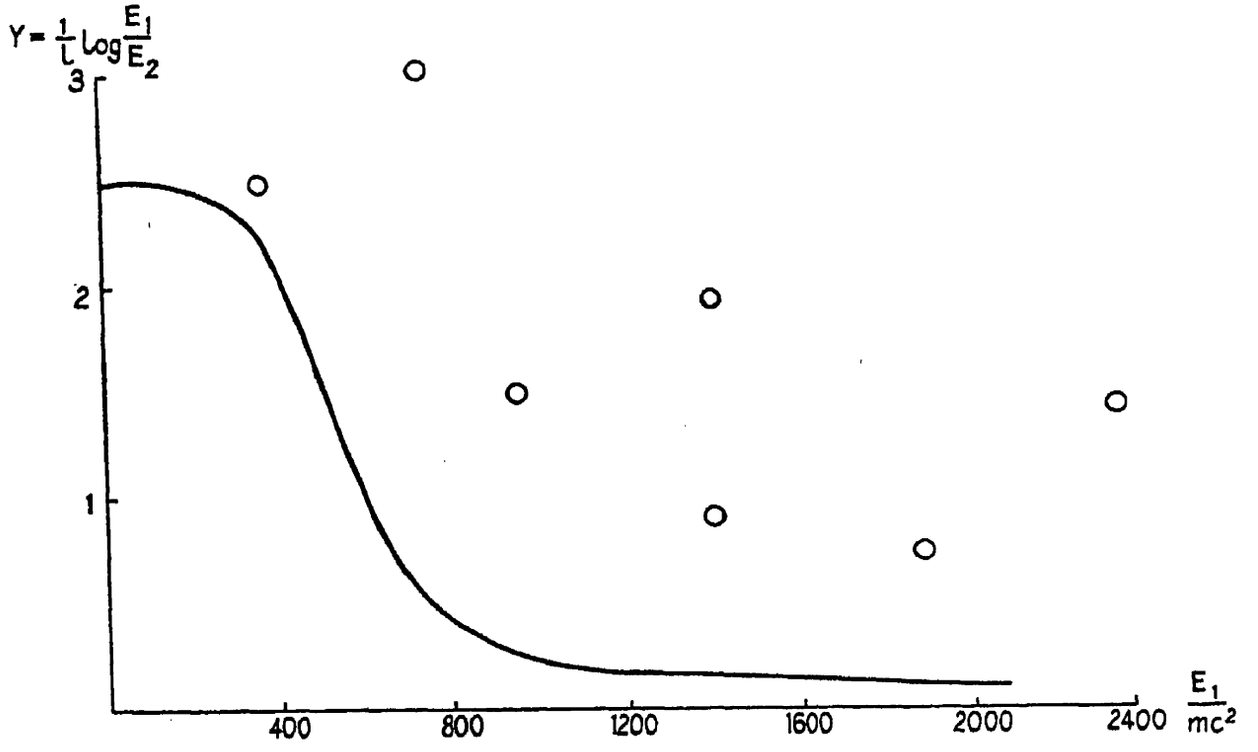

FIG. 25. The curve shows the measured average energy loss of cosmic ray particles (E_1 = initial energy, E_2 = energy after traversal of lead plate of thickness l cm.). Particles with $E_1 < 400mc^2$ are all electrons, those with $E_1 > 400mc^2$ nearly all mesons. A few electrons in the high energy region (circles) show the correct energy loss $\overline{Y} = 2{\cdot}5$ with large straggling.

Pb plate for higher energies. The curve represents th average value of Y obtained from a *large* number of cosmic ray particles. We first see that up to about $400mc^2$ the experiments confirm the theoretical predictions ($\overline{Y} = 2{\cdot}5$) very well.§ For higher energies, however, the curve drops down suddenly to a very small value. At first sight it would seem that this is in striking contradiction to the theory. In fact, the sudden decrease of the energy loss at high energies was known a long time and has led to the statement—which we now

† C. D. Anderson and S. H. Neddermeyer, *Phys. Rev.* **50** (1936), 263. In this paper earlier, inaccurate, measurements were corrected.

‡ P. M. S. Blackett, *Proc. Roy. Soc.* **165** (1938), 11; J. G. Wilson, ibid. **166** (1938), 482.

§ In representing the data corrections have been made to account for the loss by ionization.

in nature. It not only plays a predominant role in cosmic radiation, but is also the essential ingredient for our understanding of nuclei. This book, however, is not the place to deal with these questions.

6. *Annihilation probability and range of positive electrons.* Finally, we shall discuss shortly the question as to whether our calculations on the energy loss and the range of electrons have to be modified for positive electrons. This is obviously not the case in the region where Born's approximation can be applied. The transition probability is here proportional to the square of the interaction of the electron with the other particles and the sign of the charge does not therefore make any difference at all.

The only difference is that a positive electron can be *annihilated* while passing through matter. In § **21** we have calculated the probability that a positive electron is annihilated in a collision with a free negative electron. If we denote the cross-section for this process† by $\phi(E')$, the probability that the positive electron is annihilated while travelling a distance dx is given by

$$w\,dx = NZ\phi(E')\,dx.$$

For E' we have to insert the energy which the positive electron has at the particular point of its path considered. Since ϕ is given as a function of E', it is convenient to introduce the probability of annihilation per energy interval dE'. This probability is given by‡

$$\frac{NZ\phi(E')}{-dE'/dx} = w(E')\,dE', \tag{20}$$

where $-dE'/dx$ represents the energy loss per cm. as calculated in subsections 1 and 2.

If we insert for $\phi(E')$ the function § **21** eq. (11) and for dE'/dx the values given in Fig. 22, we obtain for the '*differential annihilation probability*' per energy interval dE' in lead the function plotted in Fig. 26. We see that the probability of annihilation has a maximum for $E'-mc^2 \sim mc^2$. At higher energies and at smaller energies the positive electron loses its energy so quickly (according to subsections 1 and 2) that it has no time to be annihilated. $w(E')$ gives also, in connexion with the results of § **21**, the intensity distribution of the continuous annihilation radiation (excess scattering of γ-rays).

† We consider only the two-quanta annihilation.
‡ Cf. H. Bethe, *Proc. Roy. Soc.* **150** (1935), 129.

TABLE IX

Difference in the average ranges of an ordinary electron and a positive electron in lead

E_0/mc^2	1	10	100	1,000
ΔR (*cm.*)	$0{\cdot}05 \times 10^{-2}$	0·014	0·053	0·08

24. Cascade showers

The direct measurements of the energy loss of very fast electrons and their comparison with our theory of radiative energy loss (§ 23, subsection 5) may be considered as an indication of the fact that there is at least an approximate agreement between theory and experiment. In view of the very scanty experimental data available they could, however, hardly be taken as a perfect proof of this fact. As was mentioned above, a much more convincing proof was derived from the shower phenomenon, and it was in fact only after the cascade theory of showers was developed that the experiments quoted in § 23 were performed.

Showers are one of the most conspicuous and striking phenomena in cosmic radiation. It is found that in a passage of a high-energy cosmic-ray particle through matter—a metal plate of a few cm. thickness suffices—a large number of positive and negative electrons can be created. The number of particles may be anything between 2 and several hundred, and the largest showers, known as Hoffmann bursts, consist of several thousand particles. The phenomenon was discovered by Blackett and Occhialini† in Wilson chamber experiments.

1. *The mechanism of a cascade shower.* It is indeed the theory of radiative energy loss and pair creation which leads to a complete understanding of these showers as a kind of *cascade process.* We have seen in § 23 that a fast electron loses energy quickly by radiation. After having travelled a distance $1/b$ defined by eq. (14), p. 225, the energy of the electron has, on the average, decreased to e^{-1} times its initial value E_0. A glance at Fig. 14, p. 170, at once shows that a large fraction of this energy is emitted in form of light quanta with energy k comparable with E_0 itself. It is indeed quite probable that a light quantum of, say, $k = \frac{1}{2}E_0$ is emitted. Only a small

† See reference on page 177 and numerous later papers, chiefly in *Phys. Rev.* and *Proc. Roy. Soc.*

that the number of particles increases rapidly with the thickness traversed, provided that E_0 is sufficiently large.

On these lines the cascade theory of showers was developed independently and at the same time by Carlson and Oppenheimer and by Bhabha and Heitler,† using different mathematical methods. Later improvements and amplifications with regard to the physical and mathematical approximations used have been made by several authors.‡ Naturally, the working out of this mechanism is bound to be difficult and complicated and we cannot go into the details here. All the most important features of the cascade can, however, be derived replacing the actual process by a very crude model.

The cascade process takes place in all materials in the same way if the thickness of the material is measured not in cm. but in the characteristic length b^{-1} (2). However, the mean free path of a light quantum is somewhat larger (eq. (3)), so we consider the mean value of the two distances b^{-1} and τ^{-1} which is equal to about $\frac{1}{2}$ cm. Pb or 45 cm. H_2O. We denote the thickness of matter measured in these units by t. We now make the following crude assumptions: Whenever an electron passes through the thickness $t = 1$, it emits a light quantum of half its energy, retaining the other half. Whenever a light quantum has passed through $t = 1$ it is transformed into an electron pair, both electrons receiving half the energy of the light quantum. The number of electrons + light quanta at a thickness t is therefore 2^t, and each has energy $E_0/2^t$. When this energy approaches the limit E_c, a considerable fraction of energy is actually lost in form of soft quanta which no longer produce pairs. Also energy is lost by ionization. We take these facts roughly into account by assuming that further multiplication ceases when $E_0/2^t$ becomes, say, $3E_c$. The maximum number of particles + light quanta is therefore reached at a thickness

$$2^{t_m} \sim \frac{E_0}{3E_c} \quad \text{or} \quad t_m = \frac{1}{\log 2}\left(\log \frac{E_0}{E_c} - \log 3\right). \tag{4}$$

It is easily seen that at any t not too small the number of light

† J. F. Carlson and J. R. Oppenheimer, *Phys. Rev.* **51** (1936), 220; H. J. Bhabha and W. Heitler, *Proc. Roy. Soc.* **159** (1936), 432.

‡ We quote some of the more important papers: N. Arley, *Proc. Roy. Soc.* **168** (1938), 519; L. Landau and G. Rumer, ibid. **166** (1938), 213; H. Snyder, *Phys. Rev.* **53** (1938), 960; R. Serber, ibid. **54** (1938), 317; H. J. Bhabha and S. K. Chakrabarty, *Proc. Roy. Soc.* **181** (1943), 267; *Proc. Ind. Ac. Sc.* **15** (1942), 464; S. K. Chakrabarty, *Proc. Nat. Inst. Sc. India*, **8** (1942), 331.

radiation are well above $10^6 mc^2$. Electrons of such high energy would produce in 5 cm. Pb a shower consisting of 10,000–100,000 particles. These very big showers are known as Hoffmann bursts.

It is clear that light quanta of high energies initiate the same kind of shower as electrons. Any heavier particle (mesons, for instance)

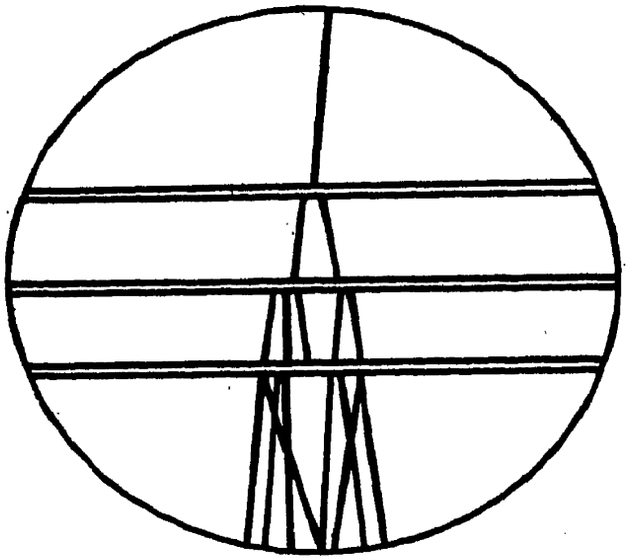

FIG. 28. Proof for the cascade nature of a shower.

would initiate a cascade shower only after having produced a fast electron by a head-on collision or emission of a high-energy light quantum. Both processes are rare compared with the rate of shower production by electrons but are known to occur.

The cascade nature of showers is put into direct evidence by the following experiment: A set of several thin lead plates with, say, 3 mm. thickness is placed across a cloud chamber and the passage of a fast electron observed. The increase of the shower size after each lead plate is then directly observed (Fig. 28). Experiments of this type have been published by many authors. They prove that showers are usually of the cascade type described here and not 'multiple processes' as had sometimes been assumed before the theory was developed. There exist, however, also showers of a different, non-cascade type. They consist probably of mesons, are initiated by protons, and have nothing in common with the electron showers discussed here.

The cascade theory of showers is in excellent agreement with all relevant experimental facts as far as the accuracy of measurements and calculations goes. We cannot enter a detailed discussion of this extensive subject here. We only draw the following, most important

this approximation for the total number of particles is small for very large E_0 but may well be 20 or 30 per cent. for the smaller values of E_0, considered in Table XI. In addition, the mathematical solution is obtained in the form of an infinite series, of which the first term gives in most cases the greatest contribution. The second term is so far evaluated only for a few data to show the error: The mathematical error caused by taking the first term only is less than 10 per cent. for all thicknesses before the shower maximum, increases to 30 per cent. after the shower maximum, and becomes 100 per cent. and more at thicknesses where the number of shower particles has become small compared with that at the maximum. The second term is always positive, so we underrate the number of particles by using the first term only.

In Table X we give the characteristic cascade units, the unit of length and the critical E_c. The unit of length $t_0 = l/t$ is defined as the reciprocal of $N\phi_{\text{rad}}$ (eq. (34), § **17**), viz.

$$t_0^{-1} = 4N\bar{\phi}\log(183Z^{-\frac{1}{2}}),$$

neglecting the small term $\frac{2}{9}$ in the definition of t_0. For the ionization loss the value is taken (and assumed to be constant) which $-(\partial E/\partial x)_{\text{coll}}$ has at the critical energy E_c, according to formula (7), § **23**, and the critical energy defined by $-(\partial E/\partial x)_{\text{coll}} = E_c N\phi_{\text{rad}}$, omitting again here the small term $\frac{2}{9}$ in ϕ_{rad}. The energy loss by collisions per one cascade unit of length is then also equal to E_c.†

TABLE X

The characteristic cascade units

	Air	H_2O	Al	Fe	Cu	Sn	Pb
Unit of length in cm.	34,000	43	9·8	1·84	1·5	1·2	0·53
Critical energy E_c in mc^2	202	225	109	51	46	24·3	13·6

In Table XI the total average number of shower particles is given as a function of the thickness t traversed and the initial energy of the electron E_0

$$y = \log\frac{E_0}{E_c}.$$

† In comparing the data by different authors it must be taken into account that the characteristic units are not always defined in exactly the same way. The definitions used in Arley's calculations differ from those of Table X by a factor log 2 for the unit of length and E_c. In Table XI his data were reduced to the above units by interpolation (accuracy of interpolation about 10 per cent.).

These formulae are very similar in their structure to those derived by our crude model, (4) and (5), and are readily understood.

For further details, especially the energy distribution of the shower particles and the fluctuations of the number of shower particles, we must refer the reader to the literature.†

* * * * * * * *

25. General formulation of a divergence-free theory including damping

Looking back at the contents of this book and asking to what extent the present quantum theory of radiation is consistent and correct, we must arrive at the following two conclusions: (i) If the interaction between the radiation field and particles is considered as small and the solution only worked out to the first non-vanishing approximation in this interaction, the theory always gives correct results which agree with the experiments up to the highest energies known. (ii) On the other hand, if the present formalism is applied to approximations higher than the first the result always diverges and the higher approximations are therefore meaningless. Strictly speaking, the present theory has no exact solutions at all and cannot therefore be regarded as an exact theory.

For certain simple problems it has been possible to find good solutions going actually somewhat beyond the first approximation. This was the case when we were dealing with questions of the line breadth (§§ 12 and 15). The solution eq. (6) of § 12, for instance, is not really the result of an expansion of the probability amplitude b according to powers of the interaction H since γ in the denominator also contains H. The damping and the line breadth arrived at in these sections are in close correspondence with the results obtained from the classical theory (§ 4) if the term $\frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{v}}$ alone is taken to describe the reaction force. It is therefore possible to treat the damping also in quantum theory at least to that extent, as it corresponds to the classical $\ddot{\mathbf{v}}$ term.

† For energies $E > E_c$ the energy spectra given by the various authors agree with each other and are correct. For $E < E_c$ or smaller the energy spectra given by Arley and by Bhabha and Chakrabarty, for instance, disagree widely, as is clear from the different approximations made. Both are equally unreliable. As yet no proper theory for this region has been given. As to the fluctuations of the number of shower particles the reader is especially referred to the paper by W. T. Scott and G. E. Uhlenbeck, *Phys. Rev.* **62** (1942), 497.

transitions will only occur between states of equal energies. The same is also true for the total momentum which is conserved even in each transition n–m. (For problems dealing with line breadth it is necessary to include also the recoil energy of the atom.) We divide all states n, m,... into two classes: states A, B, C,... with energies 'very nearly' equal to each other, $E_A \sim E_B$,..., and states i, k, α,... with energies different from E_A. The width of energy allowed for E_A will be without influence provided that it is sufficiently small, since later it will automatically tend to zero. It is our aim to derive from (I) a set of equations connecting only the probability amplitudes b_A, b_B,... with each other. We write down the equation for b_A of (I):

$$-i\hbar\dot{b}_A = H_{AB}\, b_B\, e^{i(E_B-E_A)t/\hbar} + H_{Ai}\, b_i\, e^{i(E_i-E_A)t/\hbar}, \tag{1}$$

where all the states combining with A are denoted by i, if $E_i \neq E_A$, and those for which $E_B \sim E_A$ by B. The states i in turn will combine again with other states C, $E_C \sim E_A$, and also with states α, say, of different energy. Thus

$$-i\hbar\dot{b}_i = H_{iC}\, b_C\, e^{i(E_C-E_i)t/\hbar} + H_{i\alpha}\, b_\alpha\, e^{i(E_\alpha-E_i)t/\hbar}, \tag{2}$$

and so on. The states C also include, of course, the state A. We now find solutions for b_i, b_α,.... Let E be a certain standard energy within the energy group $E_A \sim E_B \sim E_C$.... Then we put

$$b_i = X_i \frac{e^{i(E-E_i)t/\hbar}}{E-E_i}, \qquad b_\alpha = X_\alpha \frac{e^{i(E-E_\alpha)t/\hbar}}{E-E_\alpha}. \tag{3}$$

It is true that (3) does not satisfy the correct initial conditions. For if the system is at $t = 0$ in one of the states, A, B, C,..., then $b_i(0)$ should be equal to zero. However, (3) are fast periodic functions of t ($E \neq E_i$) and the amplitudes will be seen to be very small, always, because of the denominator $E-E_i$. Even if the initial conditions were satisfied at $t = 0$, after a time of only the order of magnitude of the period $\hbar/E$, b_i is excited to its full value. There can therefore be no physical meaning at all in demanding any initial conditions for b_i, b_α,.... In addition we may satisfy the initial conditions by imagining that the interaction H is switched on adiabatically at $t = 0$.

Inserting (3) into (1) and (2) we find

$$-i\hbar\dot{b}_A = H_{AB}\, b_B\, e^{i(E_B-E_A)t/\hbar} + H_{Ai}\, X_i \frac{e^{i(E-E_A)t/\hbar}}{E-E_i}, \tag{4}$$

$$X_i = H_{iC}\, b_C\, e^{i(E_C-E)t/\hbar} + H_{iA}\, b_A\, e^{i(E_A-E)t/\hbar} + \frac{H_{i\alpha}\, X_\alpha}{E-E_\alpha}, \tag{5}$$

After the second step we obtain further additional terms to (7) on the right-hand side, namely,

$$\frac{H_{Ai}H_{i\alpha}H_{\alpha D}}{(E-E_i)(E-E_\alpha)}b_D\,e^{i(E_D-E_A)t/\hbar}+\frac{H_{Ai}H_{i\alpha}H_{\alpha i'}H_{i'C}}{(E-E_i)(E-E_\alpha)(E-E_{i'})}b_C\,e^{i(E_C-E_A)t/\hbar}+$$

$$+\frac{H_{Ai}H_{i\alpha}H_{\alpha i'}H_{i'\alpha'}}{(E-E_i)(E-E_\alpha)(E-E_{i'})(E-E_{\alpha'})}X_{\alpha'}. \quad (8)$$

In the third term of (8) $X_{\alpha'}$ is still left; we need not eliminate it further.

Looking at (7) and (8) we see that the coefficients of b_C, b_D,... are the familiar matrix-elements of first, second,... and higher order, used throughout this book, remembering that E can well be put equal to E_A. We also see that it is not only the lowest-order matrix-elements for each transition that occur but also higher ones. For instance, the first term of (7) is a second-order self-energy of the state A, of the type studied in §18. The second term of (8) is a fourth-order matrix-element for the transition A–C, whilst the lowest order is the second (third term of (7)). If we were to express $X_{\alpha'}$ in (8) by (6) we would get higher and higher order matrix-elements for A–C and A–D. We call transitions of the type A–i–α–i'–C which do not lead from A to C on the shortest possible way (which is A–i–C) *round-about transitions* (see Fig. 29). (7) and (8), and what follows by extending the limiting procedure of Fig. 29 to infinity, including also states q, b,..., etc., can be written in a compact form. For this purpose we denote the matrix-elements for direct transitions A–C by $H^{(0)}_{AC}$, and those for round-about transitions of the higher order $H^{(1)}_{AC}$, $H^{(2)}_{AC}$,..., viz.

$$H^{(0)}_{AC}=\frac{H_{Ai}H_{iC}}{E-E_i}, \qquad H^{(1)}_{AC}=\frac{H_{Ai}H_{i\alpha}H_{\alpha i'}H_{i'C}}{(E-E_i)(E-E_\alpha)(E-E_{i'})}. \quad (9)$$

The coefficient of b_C in (7) is then

$$H_{AC}=H^{(0)}_{AC}+\sum_{\nu=1}^{\infty}H^{(\nu)}_{AC}. \quad (10)$$

Comprising all the states A, B, C, D, G,... by one index B over which a summation is to be carried out, we can write (7) as

$$-i\hbar\dot{b}_A=\Big(H^{(0)}_{AB}+\sum_{\nu=1}^{\infty}H^{(\nu)}_{AB}\Big)b_B\,e^{i(E_B-E_A)t/\hbar}. \quad (\mathrm{I}')$$

In the summation over B also the state A is included but, of course, $H^{(0)}_{AA}=0$ and only round-about transitions exist for A–A.

infinite field-independent polarizability of the vacuum. Our theory therefore includes also all the results concerning positive electrons derived in this book.

2. *Lorentz-invariance.* Before deriving further conclusions from (II) we show that the omission of round-about transitions does not destroy the Lorentz-invariance of the equations. We first note that (I) and therefore (I′) is invariant, being nothing but the general equations of quantum electrodynamics. We then consider the equations (10) for H_{AC}, for instance. We have seen in § **10** that every matrix-element of the nth order is proportional to the nth power of the electric charge e^n. The various terms of (10) are therefore proportional to increasing powers of e. We can therefore rewrite (10) in the form

$$H_{AC} = H^{(0)}_{AC} + \sum_{\nu=1}^{\infty} \left(\frac{e^2}{\hbar c}\right)^{\nu} \tilde{H}^{(\nu)}_{AC}, \tag{11}$$

noting that the orders of the matrix-elements obviously increase by 2. Instead of e we have taken out the dimensionless quantity $e^2/\hbar c$. All the $\tilde{H}^{(\nu)}_{AC}$ are now proportional to the same power of e as $H^{(0)}_{AC}$, and they have the same physical dimensions. (11) can then be regarded as the expansion of H_{AC} into a power series of $e^2/\hbar c$. The parameter $e^2/\hbar c$ is an invariant. Now H_{AC} will behave as some tensor component with regard to Lorentz-transformations. (In fact it is the fourth component of a 4-vector.) It is a well-known fact that if a tensor component t_{ik}, say, is expanded according to an invariant parameter

$$t_{ik} = \sum_{n=0}^{\infty} \lambda^n t^{(n)}_{ik},$$

all $t^{(n)}_{ik}$ are tensor-components of the same kind. It follows therefore that all $H^{(\nu)}_{AC}$ and $H^{(0)}_{AC}$ have the same properties with regards to Lorentz-transformations as H_{AC} itself. If therefore (I′) is Lorentz-invariant, then also (II) is invariant.

A special consideration is needed to show that the omission of the static self-energy, which was already carried out in (I), is also Lorentz-invariant. At first sight this would seem not to be the case as the static Coulomb field itself is not invariant. We must remember, however, that the separation of the electromagnetic field into a static part and a radiation field (§ **6**) is itself not an invariant procedure and had to be carried out in each Lorentz-system separately. Indeed, the static field is in *each* Lorentz-system the *instantaneous*

The equation (II) applied for b_O itself becomes

$$\gamma = \frac{2\pi}{\hbar} H_{OA} \rho_A U_{AO}. \tag{14}$$

To obtain the transition probability per unit time from O to A we form

$$\gamma_A = \lim_{t \to 0} \frac{1}{t} |b_A(t)|^2.$$

We find from (13), by integrating again over a small range of energy, dE_A,†

$$\gamma_A = \frac{2\pi}{\hbar} \rho_A |U_{AO}|^2. \tag{15}$$

γ itself is in general complex because, by (III), U_{AO} is not the complex conjugate to H_{OA}. But the real part of γ is equal to the sum of all transition probabilities from O. This follows at once from (III) and (14), remembering that $H_{OA} = H^*_{AO}$ because H_{AB} is hermitian. Thus

$$\sum \gamma_A = R(\gamma). \tag{16}$$

Thus $R(\gamma)$ is the line breadth of the initial state.

To find the transition probability we have therefore to find the U_{AO}, which are determined by (III). In the second term of (III) a summation over the states B is still to be carried out, which includes integrations of the directions of light quanta, summations over polarizations, etc. Thus (III) in general represents a system of *integral equations* for the U_{AO}. In the following subsection we shall solve (III) for the case of scattering of light by a free electron.

Eq. (15) has a great similarity to the formulae used throughout this book for the transition probabilities in first approximation. In these formulae U_{AO} of (15) is replaced by H_{AO}. This amounts to neglecting the second term of (III). We therefore conclude that the second term of (III) accounts for the damping and the reaction forces in a general way. Of course, part of the damping is already included in the γ-factors describing the time dependence of b_O and b_A. In the simple case of emission of light by an atom with O the state of the excited atom and A the state with a light quantum emitted, the matrix-elements H_{AB} can only be those of second order, describing the scattering of a light quantum, and may well be neglected. We then have $U_{AO} = H_{AO}$ and the whole damping is included in γ.

† If we are interested in questions of line breadth we must not, however, extend this integration over the line breadth, but must proceed as in § 12 and § 15. (15) gives the total transition probability, integrated over the whole line breadth.

damping. It is now very easy to see that we obtain exactly the formula (21) also in the classical theory if we include the damping force $\frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{v}}$ in the equation of motion of the electron eq. (1), § 5. We therefore conclude that *our theory contains the damping to the same extent as it is classically described by the* $\ddot{\mathbf{v}}$*-term.* One can even go farther and show in special cases that the diverging terms which we had omitted, namely, those corresponding to round-about transitions, correspond classically to those parts of the reaction force which depend on the structure of the electron and are therefore meaningless. Indeed the expansion (10) corresponds classically directly to Lorentz's expansion of the reaction force. But we cannot go into a detailed discussion of this question here.

The formula (21) has not really any scope of application. For κ becomes comparable to unity only if $k/\mu \sim \hbar c/e^2 = 137$, i.e. at such high energies that a relativistic treatment becomes imperative, which is already the case at $k \sim \mu$. (21) suggests that a modification of the Klein-Nishina formula would arise from the damping at energies $\hbar\nu > 137\mu$. We shall see, however, that this is not the case, and that the influence of damping remains negligible at all energies. To show this we have to treat the problem relativistically.

The solution of the integral equation (III) for the case of relativistic scattering is extremely difficult. We shall not attempt an exact solution but confine ourselves to show that the effect is negligible.† If the effect of damping is small we can expand

$$U_{kk_0} = H_{kk_0} + U'_{kk_0}$$

and assume that U' is small compared with H. We can then replace (18) in first approximation by

$$U_{kk_0} = H_{kk_0} + i\pi H_{kk'}\rho_{k'}H_{k'k_0}. \tag{22}$$

We have to compare the order of magnitude of the second term of (22) with that of the first term. It is convenient to do this not in the Lorentz-system where the electron is initially at rest, but in a system where the total momentum is zero: $\mathbf{p}_0 = -\mathbf{k}_0$, $\mathbf{p} = -\mathbf{k}$. The scattering process then consists only in a change of direction of $\mathbf{k}_0$, $\mathbf{p}_0$ into $\mathbf{k}$, $\mathbf{p}$, without a change of frequency: $k_0 = k$; $p_0 = p$. Since

† Wilson (loc. cit.) has made this plausible by replacing the matrix-elements in (18) by their average over the angles in a Lorentz-system where the dependence of H_{kk_0} on the angle $(\mathbf{kk}_0)$ is not very pronounced.

high energies. We could hardly expect that the theory of cascade showers could account for the large showers or Hoffmann bursts, if the damping would cut down all cross-sections at some high energy. The good agreement found can be regarded as a support of the theory of this section which *explains* that no damping is to be expected.

5. *The principle of detailed balance.* If we neglect the damping the transition probability from a state A to a state B is, apart from the density functions ρ_A, ρ_B, equal to the transition probability from B to A. This follows immediately from the fact that H_{AB}, even if it is a matrix-element of higher order such as (9), is a hermitian matrix $H^*_{BA} = H_{AB}$. The states A, B may be *completely* specified and may include the description of spin polarization and directions of all quanta and particles concerned. The statement $|H_{AB}|^2 = |H_{BA}|^2$ is known as the principle of detailed balance. It implies for instance: The probability for a light quantum $\mathbf{k}_0$ and polarization $\mathbf{e}_0$ to be scattered by an electron with initial spin $\boldsymbol{\sigma}_0$ and momentum $\mathbf{p}_0$ into a light quantum $\mathbf{k}$, polarization $\mathbf{e}$ with the electron having afterwards spin direction $\boldsymbol{\sigma}$, momentum $\mathbf{p}$, is the same as the probability for the transition $(\mathbf{k}, \mathbf{e}, \boldsymbol{\sigma}, \mathbf{p}) \to (\mathbf{k}_0, \mathbf{e}_0, \boldsymbol{\sigma}_0, \mathbf{p}_0)$.†

In the present theory this principle of detailed balance can no longer be expected to hold in its general form, as a result of the damping. The transition probability is now proportional to $|U_{AB}|^2$ and, as is seen from (III), $U_{BA} \neq U^*_{AB}$. The departure from the principle of detailed balance seems, however, not to be a very drastic one. The question has been studied for a number of examples by Hamilton and Peng‡ and it turned out that the principle always holds again whenever the transition probabilities are summed over the spin directions. A general proof for this fact has not been given, but it is likely that the departure from the principle of detailed balance, though theoretically important, will have no practical consequences.

It may very well be that the non-validity of the principle, which is entirely due to the damping, will be an essential feature of future quantum electrodynamics.

† The principle of detailed balance is to be distinguished from the reversibility of time. The reversal of the sign of t implies also a change of the direction of $\mathbf{k}$ and $\mathbf{p}$.

‡ J. Hamilton and H. W. Peng, *Proc. Roy. Ir. Ac.* **49 A** (1944), 197. The cases investigated in this paper are those arising from the meson theory rather than from radiation theory.

(iii) *The quantum-theory of vacuum fields of Born and Peng.*† Unlike (i) and (ii), the starting-point of the theory is *vacuum*-electrodynamics. The field quantities are no longer regarded as functions of the space coordinates but 'just as operators' satisfying certain commutation relations. The theory is certainly very interesting but at the time of writing not developed far enough to enable any judgement to be passed. An interesting point is that also non-linear fields can be quantized.

Against (i) and (ii) it may be objected that it is hardly likely that any classical theory of the electron can be the basis of the desired theory. For quantum electrodynamics is essentially a problem of the three universal constants, e, $\hbar$, c (relativistic quantum theory of the interaction with elementary particles). The small value of $e^2/\hbar c = 1/137$ suggests that no physical meaning can be attached to the transition $\hbar \to 0$ (c, e, finite). The starting-point of quantum electrodynamics must obviously be either non-relativistic quantum-theory of the electron ($c \to \infty$) or the quantum-theory of vacuum fields ($e \to 0$). The starting-point of (iii) is more promising.

On the whole, none of these three attempts is final or can claim as yet any conspicuous success. We therefore refrain from a more detailed discussion in this book and refer the reader to the literature. It is, in the author's opinion, not likely that the problem will be solved within the framework of radiation theory alone. As has been seen throughout this book, the reaction of the field on the particles is here always a small effect and very little guidance can be obtained from the phenomena regarding this crucial problem This is not the case in the theory of the *meson*, where the damping and reaction forces are very large. It is in processes concerning mesons where, therefore, the problems of the elementary particles and field quantization exhibit themselves (and where also the chief applications of the theory developed in § **25** lie). A discussion of these problems lies outside the framework of this book.

† M. Born and H. W. Peng, *Proc. Roy. Soc., Edinburgh*, **62** (1944), part 1, 40.

ρ charge density.

ρ_E, ρ_k density function for a light quantum (particle).
= number of states per cm.^{3} and per energy interval dk (dE).

R average range of particles.

$\mathbf{r}, \mathbf{R}$ radius vector from a centre.

$d\sigma$ surface element.

$\mathbf{S}$ Poynting vector.

S energy radiated per sec.

s spin variable.

$\sum (\sum^p)$ summation over all states of a free electron with momentum p.

S summation over spin directions (or polarization).

T kinetic energy of free particle (in Chap. I including rest energy, in Chaps. III–V excluding rest energy).

T interval of time.

τ proper time.

τ absorption coefficient for γ-rays (cm.$^{-1}$).

$d\tau$ volume element.

U, u energy (density) of field.

$\mathbf{u}$ 'kinetic momentum', $\mathbf{u} = \mathbf{p} - e\mathbf{A}$ (dimension energy). For $A = 0$ the momentum is usually denoted by $\mathbf{p}$.

u Dirac amplitude of wave function of a free electron (4 components).

v velocity.

V Coulomb interaction between two particles.

w transition probability per sec.

ϕ scalar potential, ϕ_σ Fourier component of ϕ.

ϕ cross-section ($d\phi$ differential cross-section, depending on angles).

$\mathbf{x}$ displacement of a particle.

$\mathbf{Z}$ Hertzian vector.

Z nuclear charge.

z charge of colliding particle.

Ω solid angle.

INDICES, ETC.

In general the suffix 0 refers to the initial state, no suffix to the final state, dashes to intermediate states.

A initial
F final
I, II... intermediate
} states.

'classical' means $\hbar \to 0$.

N.R. non-relativistic case (energies concerned $\ll mc^2$).

E.R. extreme relativistic case (energies concerned $\gg mc^2$).

All 'momenta' have the dimensions of energy ($c \times$ usual momentum).

UNIVERSAL LENGTHS

$$a_0 = \hbar^2/me^2 = 0{\cdot}528 \times 10^{-8} \text{ cm. (radius of hydrogen atom).}$$
$$\lambda_0 = \hbar/mc = 3{\cdot}85 \times 10^{-11} \text{ cm. (Compton wave-length).}$$
$$r_0 = e^2/mc^2 = 2{\cdot}80 \times 10^{-13} \text{ cm. (classical electronic radius).}$$
$$a_0 = 137\lambda_0 = 137^2\, r_0.$$
$$\text{'137'} = \hbar c/e^2 = 137{\cdot}3.$$

APPENDIX II

(Additional notes and remarks)

1. Angular momentum of light

In § 7, p. 63, we have discussed shortly the question of the angular momentum of light. In the case of a spherical wave emitted by an electric or magnetic multipole, the angular momentum of the wave behaves just like the angular momentum of an electron in a central field of force without spin (eq. (22), p. 63). We have not, however, discussed the important question whether a *plane wave* can have an angular momentum about the axis of propagation.

From the definition of the angular momentum (eq. (22), p. 63) it seems at first sight that this is not the case for an infinite plane wave, because the Poynting vector is then always in the direction of propagation and [**RS**] vanishes. This is not true, however, for a wave of finite extension. We consider for instance a wave extending over the volume of a cylinder travelling in the direction of the axis (z-direction) of the cylinder. Inside the cylinder the wave is to be a plane wave, the intensity dropping to zero at the wall. Then it can be shown that in general a certain angular momentum about the axis of propagation arises from the *wall* of the cylinder. The value of the total angular momentum is not, however, proportional to the surface but to the total energy U of the wave. If the wave is *circularly polarized* M_z has the value

$$M_z = \pm U/\nu, \tag{1}$$

where the sign depends on the sense of the polarization. In the quantum theory this result can be interpreted as meaning that each light quantum ($U = n\hbar\nu$) gives a contribution $\pm 1 \times \hbar$ to the angular momentum.†

If such a wave is absorbed by a screen, the angular momentum (1) will be transferred to the screen, and the latter will begin to rotate. The same is true if an infinite plane wave is absorbed by a finite screen. The angular momentum is then contained in the diffraction wave arising from the edge of the screen. The angular momentum transferred to the screen is also given by (1). The effect is essentially classical. It seems to have been observed by Beth.‡

2. Cross-section for pair production

In § 20 we have calculated the cross-section for pair production by γ-rays in the field of a nucleus. In these calculations we have made use of Born's approximation, which does not seem to be quite justifiable for heavy elements. Hulme and Jaeger|| have calculated numerically the cross-section ϕ_{pair} in lead for two energies $\hbar\nu = 3mc^2$ and $5{\cdot}2mc^2$, using the exact relativistic wave functions for the electrons.

† P. S. Epstein, *Ann. d. Phys.* **44** (1914), 593; C. G. Darwin, *Proc. Roy. Soc.* **136** (1932), 36; W. Heitler, *Proc. Camb. Phil. Soc.* **37** (1936), 112; A. W. Conway, *Proc. Roy. Ir. Ac.* **41 A** (1932), 8.

‡ R. A. Beth, *Phys. Rev.* **50** (1936), 115.

|| H. R. Hulme and J. C. Jaeger, *Proc. Roy. Soc.* **153** (1936), 443.

the longitudinal part of the right-hand side of (4a) does not contribute to (6). Naturally then, no equations for the amplitudes of the longitudinal field occur at all.

(6) and the equations of motion for the particles can be derived as Hamiltonian equations from the Hamiltonian

$$H = \sum_\lambda H_\lambda + \sum_k H_k, \qquad H_k = e_k \phi_0(k) + \sqrt{[\mu_k^2 + \{\mathbf{p}_k - e_k \mathbf{A}_1(k)\}^2]} \tag{7}$$

in the same way as in § 6. Here $\phi_0(k)$ is a function of the coordinates of the particles only. (If an external field ϕ^e, $\mathbf{A}^e$ is present, this has to be added to the potentials in (7).)

Since the only variables which occur now are the q_λ and the variables of the particles, the question of the quantization of the longitudinal field (§ 7 subsection 4) does not arise here at all. The disadvantage of this method is that the representation of the field does not appear in relativistic form, but the theory is, of course, invariant. From the quantum-mechanical point of view the gauge transformation can be regarded as a canonical transformation.

4. Action principle and canonical commutation relations

The quantization of the radiation field is based on the fact that canonical pairs of variables could be found to describe the amplitudes of each radiation oscillator (§ 6). From the point of view of general principles it is, however, desirable to find commutation relations between the field quantities themselves without referring always explicitly to the expansion into plane waves. Actually, the commutation relations between **E** and **H** (§ 8) are such relations between the operators representing **E** and **H** at each space-time point. As we shall see, **E** and the vector potential **A** at each point in space and at the same time t are canonical pairs themselves.

The formalism of canonical field variables has now become so fundamental for all quantized field theories that a short account will be given here. To find canonical pairs of variables we start, as in the dynamics of a mechanical system, from an action principle, i.e. from a Lagrangian function L. As in dynamics L is a relativistic invariant, which ensures the invariance of the subsequent theory. We confine ourselves to a pure radiation field, the inclusion of particles and their interaction with the field presents then no difficulty. We assume that L depends quadratically on the field strengths, viz.

$$L = -\frac{1}{16\pi} f_{ik}^2 = \frac{1}{8\pi}(E^2 - H^2). \tag{1}$$

Furthermore, we express **E** and **H** by the vector potential **A** which we assume to be gauged so that $\phi = 0$, $\operatorname{div} \mathbf{A} = 0$ (see item 3 of this appendix and § 6):

$$L = \frac{1}{8\pi}\left(\frac{1}{c^2}\dot{\mathbf{A}}^2 - \operatorname{curl}^2 \mathbf{A}\right), \qquad \operatorname{div} \mathbf{A} = 0. \tag{2}$$

Now **A** is not only a function of t (as every dynamical variable is) but also of the coordinates. If we are to consider the field as described by a set of dynamical variables we must consider **A** at each space point as one variable and we have therefore to deal with an infinite number of variables. For the total Lagrangian we should therefore take the sum, i.e. the integral, over all

on p. 65), namely, $(\operatorname{div}\mathbf{A})\cdot\Psi = 0$. This procedure is not very satisfactory as it puts part of Maxwell's equation (namely, $\operatorname{div}\mathbf{E} = 0$ which follows only from $\operatorname{div}\mathbf{A} = 0$) on a different footing from the rest. $\operatorname{div}\mathbf{E} = 0$ would not be satisfied as an equation between operators but only in the sense that only wave functions are to be chosen for which the eigenvalues of $\operatorname{div}\mathbf{E}$ are zero.

A more satisfactory procedure is to modify the commutation relations (6). We do that by adding a further term to the right-hand side

$$[E_i(\mathbf{r}_1)A_k(\mathbf{r}_2)] = 4\pi c i\hbar\left\{\delta_{ik}\,\delta(\mathbf{r}) - \frac{\partial^2}{\partial x_{i1}\,\partial x_{k2}}\frac{1}{4\pi r}\right\}. \tag{7}$$

If we take into account that $\nabla^2\frac{1}{r} = -4\pi\delta(\mathbf{r})$ and that $\partial/\partial x_{i1} = -\partial/\partial x_{i2}$ (because $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$) we see at once that $[E_i \operatorname{div}\mathbf{A}] = 0$. In fact (7) can quite easily be derived from the expansion into plane waves in a similar way to that in §8. From (7) the commutation relations for the field strengths (taken at the same time!) can be derived by further differentiation, the additional term of (7) does not contribute then. We find

$$[E_i\,E_k] = 0, \qquad [E_x\,H_y] = 4\pi i c\hbar\frac{\partial}{\partial z_2}\delta(\mathbf{r}).$$

These are, of course, identical with the relations (10), §8 for $t = 0$. To see this we remark that $\Delta(r, 0) = 0$ and

$$\left.\frac{\partial\Delta}{\partial t_1}\right|_{t=0} = -\left.\frac{\partial\Delta}{\partial t}\right|_{t=0} = -\frac{2c}{r}\delta'(r) = 4\pi c\,\delta(\mathbf{r}), \tag{8}$$

the latter equation being easily verified by integrating over a small volume comprising the point $r = 0$. (Note that $\int_0^\infty \delta(x)\,dx = \frac{1}{2}$.)

Finally, we note that the commutation-relations (7) can be generalized to give the commutator of two components of A at different space-time points P. We only give the result. For this purpose we introduce the function

$$h(x) = \int_0^x d\eta \int_0^\eta \delta(\xi)\,d\xi.$$

$h(x)$ has the value x for all values of $x \geqslant 0$ including $x = 0$, yet it differs from the function x in that the derivative is discontinuous at $x = 0$, namely, $h' = \frac{1}{2}$ (for $x > 0$) and $h' = 0$ for $x = 0$. We define now, in analogy to the Δ-function,

$$H(r, t) = \frac{1}{r}\{h(r+ct) - h(r-ct)\}. \tag{9}$$

The H-function has the property

$$\frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = \nabla^2 H = \Delta(r, t). \tag{10}$$

The commutation relations are then

$$[A_i(P_1)A_k(P_2)] = -i\hbar c\left\{\delta_{ik}\,\Delta(r, t) + \frac{\partial^2}{\partial x_{i1}\,\partial x_{k2}}H(r, t)\right\}. \tag{11}$$

(P denotes a space-time point.)

By differentiation all the commutation relations (10), §8 can be derived.

where $H_1^{(1)}$ is the Hankel-function. One is usually not interested in any particular impact parameter. We therefore integrate (2) over the area perpendicular to x. However, the integration must not be extended over all impact parameters but only over b larger than a certain b_m. This is a limitation imposed by quantum theory. In order to apply the method as described above, the relative sideways velocity of the two particles concerned must be small compared with c, otherwise we are not justified in treating the motion of the fast particle as a straight line and speak of a definite impact parameter b and no definite Lorentz-system could be affixed to the particle at rest. Thus by the uncertainty relation we find, as in § **17**,

$$b_m = \hbar/mc, \tag{3}$$

where m is the mass of the *lighter* of the two particles. Integrating now (2) over the area $b > b_m$, we find for the total number of equivalent light quanta

$$q(k)\,dk = 2\pi\,dk \int_{b_m}^{\infty} p(k)\,b\,db = \frac{\pi}{2}\frac{Z^2e^2}{\hbar c}\frac{dk}{k}P(z_m), \tag{4}$$

where now
$$P(z_m) = -\tfrac{1}{2}z_m^2(H_1^{(1)2}+H_0^{(1)2}) - iz_m H_0^{(1)}H_1^{(1)}. \tag{5}$$

The argument of the Hankel-functions in (5) is always iz_m. $P(z_m)$ is always positive. For $z_m \ll 1$, P becomes simply

$$P(z_m) = \frac{4}{\pi^2}(\log mc^2\gamma/k - 0{\cdot}39), \qquad \frac{k}{\gamma mc^2} \ll 1 \tag{6}$$

but decreases rapidly at $z_m \sim 1$. The condition (1) $k \ll E$ is automatically fulfilled by the rapid decrease of P and it is therefore probable that the cases where k is comparable to E and where our method fails, have a very small probability.†

If we wish to know, for instance, the radiation emitted in a collision between two electrons—a process very difficult to treat by exact methods—we only have to multiply (4) by the cross-section for scattering (§ **16**).

If we express in the Klein-Nishina formula (51), § **16**, the angle of scattering by the frequency of the scattered quantum k' we find for $k' \gg \mu$ (the term $\sin^2\theta$ is then negligible)

$$d\phi = \pi r_0^2 \mu \frac{dk'}{k^2}\left(\frac{k'}{k} + \frac{k}{k'}\right).$$

Now for the emission of a quantum k' all 'equivalent quanta' with $k > k'$ contribute. Thus the differential cross-section for the emission of a quantum k' in a collision of two electrons becomes

$$\phi\,dk' = \pi r_0^2 \mu\,dk' \int_{k'}^{\infty} q(k)\,dk\left(\frac{k'}{k^3} + \frac{1}{kk'}\right). \tag{7}$$

† This does not contradict the fact that for the radiation emitted by an electron in a passage through an atom the case $k \sim E$ has a large probability, because the above statement refers to the Lorentz-system where the nucleus is the fast-moving particle. In fact, the formulae of § **17** can, for high energies, be derived with great accuracy from (4). The bulk of the effect is always obtained if we work in the Lorentz-system where the heavier particle moves (see further below).

Similarly we find

$$q(k^*)\,dk^* \to \frac{2}{\pi 137}\,\frac{dk}{k}\left[\log\frac{\mu}{2k} - 0{\cdot}39\right]. \tag{12}$$

The integration over k is now to be performed between the limits

$$k' \quad \text{and} \quad \frac{\mu^2 k'}{4E(E-k')}.$$

Since the product of (11) and (12) is proportional to dk/k^2 only the upper limit (which is much smaller than the lower limit) gives an appreciable contribution, provided that $E-k' \gg \mu^2/4E$, which we may well assume to be the case. The condition (10) is then also automatically fulfilled. Multiplying (11) by (12) and carrying out the integration over k we obtain, writing $E-k' \equiv E'$:

$$\phi\,dk' = \frac{4r_0^2}{137}\,\frac{dk'}{k'}\,\frac{E'}{E}\left\{\left(\log\frac{2EE'}{\mu k'} - 1{\cdot}4\right)\left(\frac{E}{E'} + \frac{E'}{E} - \frac{2}{3}\right) - \frac{1}{9}\right\} \qquad (k' \gg \mu). \tag{13}$$

Here k' is no longer restricted to $k' \ll E$. (13) is larger by a factor k'/μ than the contribution (8) arising from the emission of a light quantum by the particle at rest.

(13) is almost identical with the formula (20), § 17, p. 168, for $Z = 1$, derived by the exact method. In § 17 we had, however, assumed that the particle at rest is a heavy particle and cannot take up energy. Our considerations thus show that the possibility of an energy transfer to the particle at rest does not lead to a large change in the formulae for Bremsstrahlung, even if both particles are electrons—in fact, the cases where a large amount of energy is transferred to the particle at rest are rare (at least for $k' \gg \mu$). This is, of course, all the more true, if the particle at rest is heavy.

The present method can be applied to numerous other cases which are difficult to treat by exact methods. For these we refer the reader to the paper by Williams.†

† E. J. Williams, *Kgl. Dansk. Vid. Selsk.* **13** (1935), 4.

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